Robust Estimation of a Single Complex Sinusoid in White Noise— H_∞ Filtering Approach

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Abstract—A novel robust estimator is proposed for extracting a single complex sinusoid and its parameter (frequency) from measurements corrupted by white noise. This estimator is called an H_∞ sinusoidal estimator (HSE), which is derived by applying an H_∞ filter to a noisy sinusoidal model with the state-space representation. Simulations demonstrate that the HSE is more robust to the nature of observation noise $\{v_k\}$ than the Kalman sinusoidal estimator (KSE), which is an improved version of the nonlinear filter previously proposed by the author.

Index Terms—Frequency estimation, H_∞ filter, Kalman filter, nonlinear filter, robust estimation, sinusoid.

I. INTRODUCTION

Estimation of the unknown parameters of sinusoids in noise is of great importance in many applications. Various estimation methods [1] have been proposed in each field. Linear-prediction (LP) methods are widely known. However, the location of the estimated spectral peaks can be greatly affected by a small amount of noise because of the appearance of outliers. Tufts and Kumaresan [2] dramatically improved the frequency estimation accuracy at low signal-to-noise ratio (SNR) by using a singular value decomposition (SVD) of the LP data matrix. On the other hand, the Kalman filter (KF) [3], [4] has often been applied to frequency estimation. Bitmead *et al.* [5] proposed the Kalman filtering short-time Fourier analysis, which estimates the amplitudes and phases of frequency components instead of fixing the frequencies such as $\omega_i = 2\pi f_i = 2\pi i/(N\Delta t)$. This method could be adopted for neither noise rejection nor accurate frequency estimation since the observation noise no longer remains white due to the modeling error. Anderson *et al.* [6] and Hilands *et al.* [7] have applied the extended Kalman filter (EKF) to the problem of estimating the parameters of a multiharmonic signal in white noise, where the system dynamics is linear, but the observation of the state is nonlinear, and moreover, its nonlinearity is strong, i.e., the Taylor series expansion of the observation function has infinite number of terms. The nonlinearity has often caused the EKF to diverge in some poor initial conditions. To overcome these problems, a guaranteed-stability nonlinear filter based on EKF (precisely ECKF) has been proposed for estimating a single complex sinusoid in white noise [8]. In general, however, the performance of the Kalman filter is degraded under non-Gaussian white noises since the Kalman filter is sensitive to the Gaussian property of noises.

In this correspondence, a robust estimator known as an H_∞ sinusoidal estimator (HSE) is proposed for simultaneously estimating both a single complex sinusoid and its frequency (precisely a function of frequency) from measurements corrupted by Gaussian or non-Gaussian white noises.

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II. PROBLEM FORMULATION

Let an observation signal y_k at time t_k be a single sinusoid z_k with the additive noise v_k

$$y_k = z_k + v_k, \quad k = 1, 2, 3, \dots, N \quad (1)$$

where $z_k = a_1 \exp(j\omega_1 t_k)$, $a_1 = |a_1|e^{j\phi_1}$, $\omega_1 = 2\pi f_1$, $t_k = k\Delta t$ in which a_1 and f_1 are a complex amplitude with initial phase ϕ_1 and the frequency of the sinusoid, respectively, Δt is a sampling interval, and $j^2 = -1$. The observation noise v_k is a complex white noise with zero-mean, and in general cases, the real and imaginary parts are correlated as

$$E\left\{\begin{bmatrix} v_k^r \\ v_k^i \end{bmatrix} \begin{bmatrix} v_k^r \\ v_k^i \end{bmatrix}^T\right\} = \begin{bmatrix} \sigma_{v_r}^2 & s_v \\ s_v & \sigma_{v_i}^2 \end{bmatrix}.$$

Our goal here is to estimate a single complex sinusoid $\{z_k\}$ and its frequency f_1 in the presence of white noise, where the complex amplitude a_1 and the covariance $\begin{bmatrix} \sigma_{v_r}^2 & s_v \\ s_v & \sigma_{v_i}^2 \end{bmatrix}$ are assumed to be previously unknown.

III. H_∞ SINUSOIDAL ESTIMATOR (HSE)

A. Preparation

Before deriving the H_∞ sinusoidal estimator (HSE), we shall introduce the fundamentals of the H_∞ filtering theory [9], [10].

1) *H_∞ Filtering Problem:* Consider a time-variant state-space model of the form

$$\begin{aligned} x_{k+1} &= F_k x_k + G_k w_k \\ y_k &= H_k x_k + v_k, \quad k = 0, 1, 2, \dots \end{aligned} \quad (2)$$

where F_k, G_k, H_k are known matrices, $x_0, \{w_k\}, \{v_k\}$ are unknown quantities regarded as disturbances, and $\{y_k\}$ is the observation. Here, the noise processes have the bounded l_2 norm, i.e., $\|w\|_2^2 = \sum_{k=0}^{\infty} w_k^* w_k < \infty$, $\|v\|_2^2 = \sum_{k=0}^{\infty} v_k^* v_k < \infty$, but the Gaussian of the noise processes is not assumed.

In such a situation, we would like to estimate an arbitrary linear combination of the state x_k , say, $z_k = L_k x_k$, using the observations $\{y_k\}$, where L_k is given. All the above equations can be also summarized as

$$\begin{bmatrix} x_{k+1} \\ y_k \\ z_k \end{bmatrix} = \begin{bmatrix} F_k & 0 & B_k^1 \\ H_k & 0 & B_k^2 \\ L_k & 0 & 0 \end{bmatrix} \begin{bmatrix} x_k \\ u_k \\ d_k \end{bmatrix} \quad (3)$$

where

$$d_k = [w_k^T \ v_k^T]^T, \quad B_k^1 = [G_k \ 0], \quad B_k^2 = [0 \ I]$$

which is a form used in H_∞ control.

Let $\hat{z}_{k|k} = \mathcal{F}_f(y_0, y_1, \dots, y_k)$ denote the estimate of z_k , given observations $\{y_k\}$ from t_0 to t_k . We then have the following estimation error:

$$e_{f,k} = \hat{z}_{k|k} - z_k = \hat{z}_{k|k} - L_k x_k. \quad (4)$$

As illustrated in Fig. 1, let $T_k(\mathcal{F}_f)$ denote the transfer operator that maps the unknown disturbances $\Sigma_0^{-1/2}(x_0 - \hat{x}_0|_{-1})$, $\{w_i\}_{i=0}^k$, $\{v_i\}_{i=0}^k$ to the filtered error $\{e_{f,i}\}_{i=0}^k$, where Σ_0

Fig. 1. Transfer matrix from disturbances to filtered error.

is a positive-definite matrix that reflects *a priori* knowledge as to how close x_0 is to the initial guess $\tilde{x}_{0|-1}$.

Our object is to choose a functional $\mathcal{F}_f$ to minimize the H_∞ norm of the transfer operator $T_k(\mathcal{F}_f)$. This problem is called the H_∞ optimal filtering problem.

In general, the H_∞ norm of a transfer operator T is defined as

$$\|T\|_\infty = \sup_{u \in l_2, u \neq 0} \frac{\|Tu\|_2}{\|u\|_2} \quad (5)$$

using the input u and output $y = Tu$, where u has the bounded l_2 norm, i.e., $\|u\|_2^2 = \sum_{k=0}^\infty u_k^T u_k = \sum_{k=0}^\infty \|u_k\|^2 < \infty$, and $l_2 = \{u \mid \|u\|_2 < \infty\}$ ($\|u\|_2$ is the 2-norm of the causal sequence $\{u_k\}$). The H_∞ norm thus has the interpretation of being the maximum energy gain from the input u to the output y .

A closed-form solution of the optimal H_∞ estimation problem is available only in specific cases, and therefore, it is common in the literature to settle for a (finite-time) suboptimal solution.

Suboptimal H_∞ Filtering Problem: Given a scalar $\gamma_f > 0$, find an H_∞ suboptimal estimation strategy $\tilde{z}_{k|k} = \mathcal{F}_f(y_0, y_1, \dots, y_k)$ that achieves $\|T_k(\mathcal{F}_f)\|_\infty < \gamma_f$. In other words, find a strategy that achieves

$$\sup_{x_0, \{w_i\}, \{v_i\}} \frac{\sum_{i=0}^k \|e_{f,i}\|^2}{\|x_0 - \tilde{x}_{0|-1}\|_{\Sigma_0^{-1}}^2 + \sum_{i=0}^k \|w_i\|^2 + \sum_{i=0}^k \|v_i\|^2} < \gamma_f^2. \quad (6)$$

This clearly requires checking whether $\gamma_f \geq \gamma_{f,op}$, where $\gamma_{f,op} = \inf_{\mathcal{F}_f} \|T_k(\mathcal{F}_f)\|_\infty$.

The solution (optimal H_∞ filter) of the optimal H_∞ filtering problem can be obtained to the desired accuracy by iterating on the γ_f of the suboptimal H_∞ problem. Therefore, the solution to the suboptimal H_∞ problem is called an H_∞ filter hereafter.

2) H_∞ Filter: We now present the existing solution to the suboptimal H_∞ filtering problem.

H_∞ Filter: For a given $\gamma_f > 0$, one possible level- γ_f H_∞ filter that achieves $\|T_k(\mathcal{F}_f)\|_\infty < \gamma_f$ is given by

$$\tilde{z}_{k|k} = L_k \hat{x}_{k|k} \quad (7)$$

$$\hat{x}_{k+1|k} = F_k \hat{x}_{k|k} \quad (8)$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_{s,k}(y_k - H_k \hat{x}_{k|k-1}) \quad (9)$$

$$K_{s,k} = \hat{P}_{k|k-1} H_k^{*T} (I + H_k \hat{P}_{k|k-1} H_k^{*T})^{-1} \quad (10)$$

$$\begin{aligned} \hat{P}_{k+1|k} &= F_k \hat{P}_{k|k-1} F_k^{*T} + G_k G_k^{*T} \\ &\quad - F_k \hat{P}_{k|k-1} [H_k^{*T} \quad L_k^{*T}] R_{e,k}^{-1} \begin{bmatrix} H_k \\ L_k \end{bmatrix} \hat{P}_{k|k-1} F_k^{*T} \end{aligned}$$

where

$$\begin{aligned} R_{e,k} &= R_k + \begin{bmatrix} H_k \\ L_k \end{bmatrix} \hat{P}_{k|k-1} \begin{bmatrix} H_k^{*T} & L_k^{*T} \end{bmatrix} \\ R_k &= \begin{bmatrix} I & 0 \\ 0 & -\gamma_f^2 I \end{bmatrix}. \end{aligned} \quad (11)$$

For the existence of the H_∞ filter, $\hat{P}_{k|k-1}$ (or $\hat{P}_{k|k}$) needs to satisfy

$$\begin{aligned} \hat{P}_{k|k}^{-1} &= \hat{P}_{k|k-1}^{-1} + [H_k^{*T} \quad L_k^{*T}] \begin{bmatrix} I & 0 \\ 0 & -\gamma_f^{-2} I \end{bmatrix} \begin{bmatrix} H_k \\ L_k \end{bmatrix} \\ &= \hat{P}_{k|k-1}^{-1} + H_k^{*T} H_k - \gamma_f^{-2} L_k^{*T} L_k > 0 \end{aligned} \quad (12)$$

where the $[F_i \quad G_i]$ have full rank. The existence condition can be replaced by another condition in which R_k and $R_{e,k}$ have the same inertia, i.e.,

$$\begin{aligned} I + H_k \hat{P}_{k|k-1} H_k^{*T} &> 0 \\ -\gamma_f^2 I + L_k (\hat{P}_{k|k-1}^{-1} + H_k^{*T} H_k)^{-1} L_k^{*T} &< 0 \end{aligned} \quad (13)$$

which is used to construct the H_∞ filter.

B. Derivation of H_∞ Sinusoidal Estimator (HSE)

Using the fact that the difference equation $z_k = \alpha(1)z_{k-1}$ holds for a single complex sinusoid $\{z_k\}$ and some ideas [$\alpha(1) = \alpha(1)$ holds over time because $\alpha(1)$ is constant with respect to time], we have

$$\begin{aligned} \begin{bmatrix} \alpha(1) \\ z_{k+1} \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & \alpha(1) \end{bmatrix} \begin{bmatrix} \alpha(1) \\ z_k \end{bmatrix} \\ y_k &= [0 \quad 1] \begin{bmatrix} \alpha(1) \\ z_k \end{bmatrix} + v_k \end{aligned} \quad (14)$$

where $\alpha(1) = \exp(j\omega_1 \Delta t)$ is an unknown parameter, which is called an AR coefficient. Here, replacing $\alpha(1)$ in the dynamics by $\alpha(1) + \delta\alpha(1)$ to allow for the uncertainty $\delta\alpha(1)$ of $\alpha(1)$, the above state equation is represented by

$$\begin{bmatrix} \alpha(1) \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \alpha(1) \end{bmatrix} \begin{bmatrix} \alpha(1) \\ z_k \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_k \quad (15)$$

where the mean of $w_k = \delta\alpha(1)z_k$ is assumed to be zero for simplicity.

This linear state-space model with an unknown parameter $\alpha(1)$ can be rewritten by the following nonlinear stochastic system:

$$x_{k+1} = f(x_k) + Gw_k, \quad y_k = Hx_k + v_k \quad (16)$$

where

$$\begin{aligned} x_k &= [x_k(1) \quad x_k(2)]^T = [\alpha(1) \quad z_k]^T \\ f(x_k) &= [\alpha(1) \quad \alpha(1)z_k]^T, \quad G = [0 \quad 1]^T \\ H &= [0 \quad 1]. \end{aligned} \quad (17)$$

Linearizing the above nonlinear system (the terms higher than the first order are ignored) and then applying the H_∞ filter to the linearized system, we can obtain an H_∞ sinusoidal estimator (HSE) to provide robust estimation of the linear combination $z_k = Hx_k$ ($L_k = H$)

$$\tilde{z}_{k|k} = H \hat{x}_{k|k} \quad (18)$$

$$\hat{x}_{k+1|k} = f(\hat{x}_{k|k}) \quad (19)$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_{s,k}(y_k - H \hat{x}_{k|k-1}) \quad (20)$$

$$K_{s,k} = \hat{P}_{k|k-1} H^{*T} (1 + H \hat{P}_{k|k-1} H^{*T})^{-1} \quad (21)$$

$$\hat{P}_{k+1|k} = F_k \hat{P}_{k|k-1} F_k^{*T} + \beta^2 G G^{*T}$$

$$- F_k \hat{P}_{k|k-1} [H^{*T} \quad H^{*T}] R_{e,k}^{-1} \begin{bmatrix} H \\ H \end{bmatrix} \hat{P}_{k|k-1} F_k^{*T}$$

where

$$\begin{aligned}
 \hat{x}_{k|k} &= \begin{bmatrix} \hat{x}_{k|k}(1) \\ \hat{x}_{k|k}(2) \end{bmatrix} \\
 &= \begin{bmatrix} \hat{\alpha}_{k|k}(1) \\ \hat{z}_{k|k} \end{bmatrix} \\
 F_k &= \left. \frac{\partial f(x_k)}{\partial x_k} \right|_{x_k = \hat{x}_{k|k}} \\
 &= \begin{bmatrix} 1 & 0 \\ \hat{x}_{k|k}(2) & \hat{x}_{k|k}(1) \end{bmatrix} \\
 R_{e,k} &= R_k + \begin{bmatrix} H \\ H \end{bmatrix} \hat{P}_{k|k-1} \begin{bmatrix} H^{*T} & H^{*T} \end{bmatrix} \\
 R_k &= \begin{bmatrix} 1 & 0 \\ 0 & -\gamma_f^2 \end{bmatrix} \\
 \hat{P}_{k|k}^{-1} &= \hat{P}_{k|k-1}^{-1} + (1 - \gamma_f^{-2}) H^{*T} H > 0
 \end{aligned} \quad (22)$$

in which superscripts $*$ and T denote the complex conjugate and transpose, respectively.

IV. RELATIONSHIP WITH THE KALMAN SINUSOIDAL ESTIMATOR (KSE)

In general, the formula of the H_∞ filter is very similar to that of the Kalman filter. However, for the H_∞ filter to exist, additional conditions (12) or (13) must be satisfied, and L_k and the indefinite matrix $R_k = \begin{bmatrix} I & 0 \\ 0 & -\gamma_f^2 I \end{bmatrix}$ appear in the equations, which would not appear in the Kalman filter.

To clarify the relationship between the HSE and the KSE [8], increasing γ_f to ∞ , we have

$$\begin{aligned}
 R_{e,k}^{-1} &= \left(\begin{bmatrix} 1 & 0 \\ 0 & -\gamma_f^2 \end{bmatrix} + \begin{bmatrix} H \\ H \end{bmatrix} \hat{P}_{k|k-1} \begin{bmatrix} H^{*T} \\ H^{*T} \end{bmatrix} \right)^{-1} \\
 &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} H \\ 0 \end{bmatrix} \left(H^{*T} H + \hat{P}_{k|k-1}^{-1} \right)^{-1} \begin{bmatrix} H^{*T} & 0 \end{bmatrix}.
 \end{aligned} \quad (23)$$

Substituting the above into (21), we see that the Riccati recursion in (21) reduces to

$$\begin{aligned}
 \hat{P}_{k+1|k} &= F_k \hat{P}_{k|k-1} F_k^{*T} + \beta^2 G G^{*T} \\
 &\quad - F_k \hat{P}_{k|k-1} H^{*T} \left(1 - H (H^{*T} H + \hat{P}_{k|k-1}^{-1})^{-1} H^{*T} \right) \\
 &\quad \cdot H \hat{P}_{k|k-1} F_k^{*T}.
 \end{aligned} \quad (24)$$

Furthermore, this can be rewritten as

$$\begin{aligned}
 \hat{P}_{k+1|k} &= F_k \hat{P}_{k|k-1} F_k^{*T} + \beta^2 G G^{*T} - F_k \hat{P}_{k|k-1} H^{*T} \\
 &\quad \cdot \left(1 + H \hat{P}_{k|k-1} H^{*T} \right)^{-1} H \hat{P}_{k|k-1} F_k^{*T}
 \end{aligned} \quad (25)$$

using the inversion lemma. Hence, the HSE when $\gamma_f = \infty$ is presented by

$$\begin{aligned}
 \hat{x}_{k+1|k} &= f(\hat{x}_{k|k}) \\
 \hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_{s,k} (y_k - H \hat{x}_{k|k-1})
 \end{aligned} \quad (26)$$

$$K_{s,k} = \hat{P}_{k|k-1} H^{*T} \left(H \hat{P}_{k|k-1} H^{*T} + 1 \right)^{-1} \quad (27)$$

$$\begin{aligned}
 \hat{P}_{k+1|k} &= F_k \hat{P}_{k|k} F_k^{*T} + \beta^2 G G^{*T} \\
 \hat{P}_{k|k} &= \hat{P}_{k|k-1} - K_{s,k} H \hat{P}_{k|k-1}.
 \end{aligned} \quad (28)$$

This agrees with the formula of the KSE, which is a version with system noise of [8] obtained by applying the extended Kalman filter to (16). Consequently, we see that the HSE, when $\gamma_f = \infty$, is in a sense equivalent to the KSE.

Fig. 2. MSE and VAR characteristics of HSE ($\gamma_f = 1.2$, $\beta^2 = 0.1$) and KSE ($\beta^2 = 0.1$); complex Gaussian case of v_k , $N = 40$, $f_1 = 7.2/60$, $a_1 = e^{j\pi/6}$.

This fact suggests that the H_∞ norm concerning the KSE is quite large and that the KSE may have poor robustness properties to variations in x_0 , $\{w_k\}$, and $\{v_k\}$. This will be verified by simulations presented in next section.

V. SIMULATION STUDY

In this section, the performance of the HSE is evaluated from various points of view using computer simulations, and it is compared with that of the KSE. Now, we consider 40 data sets of the length $N = 40$ under the same SNR, which consists of a single complex sinusoid with additive white noise

$$y_k = z_k + v_k, \quad k = 1, \dots, 40 \quad (29)$$

where $z_k = a_1 e^{j\omega_1 t_k}$, $a_1 = 1 \cdot e^{j\pi/6}$, $\omega_1 = 2\pi f_1$, $f_1 = 7.2/60 (= 4.8/40)$, $t_k = k\Delta t$, and $\Delta t = 1$ is set for simplicity. The variance σ_v^2 of stationary, complex white noise v_k with zero mean is chosen to yield a given SNR, which is defined by $\text{SNR} = 10 \log_{10}(|a_1|^2 / \sigma_v^2) [\text{dB}]$.

A. Robustness of the HSE to the Observation Noise Variations

The Kalman filter is strictly optimal only if x_0 , $\{w_k\}$, and $\{v_k\}$ are Gaussian. In contrast, as seen in (6), the H_∞ filter is designed using the energies of x_0 , $\{w_k\}$, and $\{v_k\}$ and not their nature (Gaussian distributed, uncorrelated, etc.). Hence, it is expected to be robust to variations in x_0 , $\{w_k\}$, and $\{v_k\}$. First of all, to ensure the robustness of the HSE to variations in $\{v_k\}$, we focus on the performances of the HSE and the KSE in a noncomplex Gaussian case of v_k (the real and imaginary parts of v_k are correlated), where $v_k = v_k^r + j v_k^i$ is set such that

$$E \left\{ \begin{bmatrix} v_k^r \\ v_k^i \end{bmatrix} \begin{bmatrix} v_k^r & v_k^i \end{bmatrix} \right\} = \begin{bmatrix} 9\sigma_{v_i}^2 & -3\sigma_{v_i}^2 \\ -3\sigma_{v_i}^2 & \sigma_{v_i}^2 \end{bmatrix}$$

i.e., $\text{Re}\{v_k\} = -3\text{Im}\{v_k\}$. Note that such a case will happen when the complex signal is generated from a real signal using Hilbert transform. Figs. 2 and 3 show the mean square error (MSE) and the variance (VAR) characteristics [8] in complex Gaussian and correlated cases of $\{v_k\}$, respectively. Here, $\hat{x}_{1|0}$ and $\hat{P}_{1|0}$ are set as in [8]. In addition, we have to choose γ_f to be as small as possible, checking the existence condition in (22), to effectively use the H_∞ filter. However, in order to guarantee the stability of the HSE, γ_f must be selected such that $\gamma_f > 1.0$ (The stability of the HSE is proved by same way as in [8]).

These results demonstrate that the HSE is considerably more robust than the KSE to the correlation of real and imaginary parts of v_k (see Section IV).

Fig. 3. MSE and VAR characteristics of HSE ($\gamma_f = 1.2$, $\beta^2 = 0.1$) and KSE ($\beta^2 = 0.1$); correlated case of $v_k(\text{Re}\{v_k\} = -3\text{Im}\{v_k\})$, $N = 40$, $f_1 = 7.2/60$, $a_1 = e^{j\pi/6}$.

Fig. 4. MSE and VAR characteristics of HSE ($\gamma_f = 1.2$, $\beta^2 = 0.1$) and KSE ($\beta^2 = 0.1$); mixture case of v_k , $N = 40$, $f_1 = 7.2/60$, $a_1 = e^{j\pi/6}$.

Furthermore, we consider another non-Gaussian case where v_k obeys a mixture distribution such that

$$v_k = \begin{cases} v_k^{(1)}, & \text{with probability } 1 - \varepsilon \\ v_k^{(2)}, & \text{with probability } \varepsilon \end{cases} \quad (30)$$

where $v_k^{(i)}$ is complex Gaussian with zero mean and variance σ_i^2 . In this case, the probability density function of v_k is given by

$$p(v) = (1 - \varepsilon)p_1(v) + \varepsilon p_2(v) \\ = (1 - \varepsilon) \frac{1}{\pi\sigma_1} e^{-(|v|^2/\sigma_1^2)} + \varepsilon \frac{1}{\pi\sigma_2} e^{-(|v|^2/\sigma_2^2)}. \quad (31)$$

This may be a more reasonable assumption in many practical applications, which means that there are some outliers in the noise process. Fig. 4 shows the MSE and VAR characteristics in a mixture case of v_k , where $\varepsilon = 0.1$ and $\sigma_2 = 8\sigma_1$ are set so that $\sigma_v^2 = 7.3\sigma_1^2$. Note that ε and σ_1 are assumed to be unknown in the simulations.

From the above result, it is seen that the HSE is also more robust than the KSE to variations in the distribution of v_k , especially for the mixture case. Note that the Cramér-Rao bound (CRB) in Figs. 3 and 4 is calculated under the complex Gaussian assumption.

B. Robustness of the HSE to Variations in the Initial State and the Tuning Parameter β^2

From $\|T_k(\mathcal{F}_f)\|_\infty < \gamma_f$, it is expected that the HSE will be also robust to variations in the initial state and the tuning parameter β^2 (system noise) as well as the observation noise. To ensure this, we make simulations for the following two cases: 1) $\hat{x}_{1|0} = [0, \hat{z}_{1|0}]^T$, $\beta^2 = 0.1$ and 2) $\hat{x}_{1|0} = [\hat{\alpha}_{1|0}, \hat{z}_{1|0}]^T$, $\beta^2 = 0.01$, as shown in Fig. 5.

Fig. 5. MSE characteristics of HSE ($\gamma_f = 1.2$) and KSE; complex Gaussian case of v_k . (a) $\hat{x}_{1|0} = [0, \hat{z}_{1|0}]^T$, $\beta^2 = 0.1$. (b) $\hat{x}_{1|0} = [\hat{\alpha}_{1|0}, \hat{z}_{1|0}]^T$, $\beta^2 = 0.01$, where $N = 40$, $f_1 = 7.2/60$, $a_1 = e^{j\pi/6}$.

These results show that the HSE is also a little more robust than the KSE to variations in the initial state and the tuning parameter β^2 (system noise).

VI. DISCUSSION AND CONCLUSION

The robust estimator, which is called the HSE, has been proposed for estimating a single complex sinusoid and its parameter (frequency) in the presence of white noise. Some simulations showed that the HSE is more robust than the KSE to variations in the observation noise $\{v_k\}$ as well as the initial state and tuning parameter (system noise).

Some questions, however, remain unanswered. For example, it has not yet been confirmed whether the HSE is effective for multiple sinusoids. However, it will be a future work.

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