On a Period Analysis for Musical sound of Certain Musical Instrument of Percussion

By R. Hiraga.

In a musical sound of striking musical instrument, for example, Korean Stone musical instruments at Confucius Shrine in Seoul, though it is beautiful, we can not always say that in all components of a musical sound their frequencies have multiple integral numbers of their least frequency.

If we consider such a case that logarithmic decrements of component waves are different and some of them have an increasing amplitude, we can apply Fourier's Analysis for only a periodic wave with constant state.

In the experiment, as ordinarily a sound photographic image is given by the following expression:--

\[ U = \sum_{a} e^{-k_r} \sin \frac{2\pi}{T} (t + \gamma) \]  

(1)

For (1) we can apply an Analysis of E. T. Whittaker and G. Robinson in the special case of \( K = 0 \). I. Aoki used their method and tried to make acoustic spectrums with analyzed sine wave groups. K. Takahashi tried the another period analysis for the different things from sound.

O. F. Socia's mechanical method which is applied for aperiodic sound waves stands the another standpoint with us. We will propose a method of period Analysis applicable for sound wave whose components have \( K \neq 0 \).

We draw two axes on the properly enlarged photographic image of sound wave, one is parallel to the time axis of wave and the other is perpendicular to it, then denote the ordinates corresponding to the end points of the properly small equidistantly divided time intervals, as follows:--
On a Period Analysis for Musical sound of Certain Musical Instrument of Percussion By R. Hiraga.

\[ V_0, V_1, V_2, V_3, \ldots, V_i, \ldots \] 

We will use \( u_i \) for the standard deviation of \( V_i \)'s that is

\[ u_i = V_i - V_m \quad (i=0, 1, 2, 3, \ldots) \]

Where \( V_m \) = the arithmetic mean of \( V_i \)'s.

and we make a following arrangement with \((2n+1)\) lines and \(p\) columns of successive \( u_i \)'s:

\[
\begin{array}{cccccccc}
  u_0 & u_1 & u_2 & \cdots & u_N & u_{N+1} & u_{N+2} & \cdots \\
  u_p & u_{p+1} & u_{p+2} & \cdots & u_{2p-1} & u_{2p} & u_{2p+1} & \cdots \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
  u_{np} & u_{np+1} & u_{np+2} & \cdots & u_{(n+1)p-1} & u_{(n+1)p} & u_{(n+1)p+1} & \cdots \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
  u_{2np} & u_{2np+1} & u_{2np+2} & \cdots & u_{(2n+1)p-1} & u_{(2n+1)p} & u_{(2n+1)p+1} & \cdots \\
\end{array}
\]  

(2)

where \( u_{np+1} \) = central term in the arrangement.

(1) 佐藤楽二:科学 7(1937), 364. Kalähne: Grundräger der math-phys. Austik 1.kap.2


(3) I. Aoki: memoires, coll. of Sci. Kyōto, 14, 213-218(1931)

(4) 高橋浩: 理化学研究所彙報 第十四輯第四號 PP.225-271(昭和十年四月)


In (2) \star 以下本文中 \( K_i \) は \( K_i \) を示し, \( K_i \) は \( K_i \) を示す。 (印刷の都合上)

\[
\begin{align*}
  u_i &= a_i e^{-K_i^i} \sin \frac{2\pi}{T_1} (t+\gamma_i) + \sum_{k=0}^{m} a_i e^{-K_i^i} \sin \frac{2\pi}{T_1} (t+\gamma_k), \\
  \text{where} \\n  a_i e^{-K_i^i} \sin \frac{2\pi}{T_1} (t+\gamma_i) &= a \text{ Wave component with period } T_1. \\
  m &= a \text{ number of wave components contained the Sound.}
\end{align*}
\]  

--- [38] ---
Now we use following symbols, -

\[ U_i = \sum_{s=0}^{2n} a_{1e} \sin(t + \gamma_1 + a\nu) + \sum_{s=0}^{m} \sum_{i=2}^{m} a_{e} \sin \frac{2\pi}{T_1} (t + \gamma_i + a\nu) \]  \hspace{1cm} (4)

where \( \nu = \) equidevided time interval.

and

\[ M_i^2 = \frac{1}{(2n+1)^2} U_i^2 \]  \hspace{1cm} (5)

The magnitude of \( \nu \) must be taken so small as the following three approximations are satisfied.

\[
\begin{align*}
\int_{0}^{p} u_{ip+1} dt & \equiv \nu \sum_{i=0}^{p-1} u_{ip+1} \\
\int_{0}^{p} U_i^2 dt & \equiv \nu \sum_{i=0}^{p-1} U_i^2 \\
\int_{0}^{p} M_i^2 dt & \equiv \nu \sum_{i=0}^{p-1} M_i^2
\end{align*}
\]

Case (1)

A case of no integral multiple ratio between any two frequencies of components in a sound.

If there is no integral multiple ratio between any two periods of components in a sound, the following two assumptions are brought, that is, if we have no component wave whose period \( T_i \) has like \( (T_1, T_i) = T_1 \) in

\[ \sum_{i=2}^{m} a_i e^{-K(t + a\nu)} \sin \frac{2\pi}{T_1} (t + \gamma_i), \]

we have:

(i) There is no correlation between \( \sum_{s=0}^{2n} a_{1e} e^{-K(t + a\nu)} \sin \frac{2\pi}{T_1} (t + \gamma_1 + a\nu) \)

and \( \sum_{s=0}^{2n} \sum_{i=2}^{m} a_i e^{-K(t + a\nu)} \sin \frac{2\pi}{T_1} (t + \gamma_i + a\nu) \)

(ii) There is no correlation between \( \sum_{i=2}^{m} a_i e^{-K(t + a\nu)} \sin \frac{2\pi}{T_1} (t + \gamma_i + a\nu) + \sum_{s=0}^{2n} \sum_{i=2}^{m} a_i e^{-K(t + a\nu)} \sin \frac{2\pi}{T_1} (t + \gamma_i + a\nu) \)

\(- (39) -\)
On a Period Analysis for Musical sound of Certain Musical Instrument of Percussion By R. Hiraga.

where \( \alpha' = \) any number of \( \alpha \)

From (i)

\[
\frac{1}{p^\nu} \int_0^{p^\nu} M_i \, dt = \frac{1}{(2n+1)^2} \int_0^{p^\nu} \left[ \left\{ \sum_{\alpha=0}^{2n} \sum_{\nu=0}^m a_{\alpha} e^{\frac{-K(t+\nu \gamma)}{T_i}} \sin \frac{2\pi}{T_i} (t+\nu \gamma + \alpha \nu) \right\}^2 \right] \, dt
\]

From (ii)

\[
\int_0^{p^\nu} \left[ \sum_{\alpha=0}^{2n} \left( \sum_{\nu=0}^m a_{\alpha} e^{\frac{-K(t+\nu \gamma)}{T_i}} \sin \frac{2\pi}{T_i} (t+\nu \gamma + \alpha \nu) \right)^2 \right] \, dt
\]

\[
= \int_0^{p^\nu} \left[ \sum_{\alpha=0}^{2n} \left( \sum_{\nu=0}^m a_{\alpha} e^{\frac{-K(t+\nu \gamma)}{T_i}} \sin \frac{2\pi}{T_i} (t+\nu \gamma + \alpha \nu) \right)^2 \right] \, dt
\]

\[
= \sum_{\alpha=0}^{2n} \sigma^2_{\alpha}
\]

(6)

where \( \sigma_{\alpha} \) = standard deviation of \( \sum a_{\alpha} e^{\frac{-K(t+\nu \gamma)}{T_i}} (t+\nu \gamma) \)

Now, with respect to \( \frac{2n}{\sum a_{\alpha} e^{\frac{-K(t+\nu \gamma)}{T_i}} (t+\nu \gamma + \alpha \nu)} \), \( K_{\alpha} \nu \)

be so small that we can assume the following expression:–

\[
e^{\frac{-K_{\alpha} \nu}{T_i}} \left\{ \begin{array}{l}
e^{1-K_{\alpha} \nu} \quad 0 < \alpha < 2n+1 \\
e^{1+K_{\alpha} \nu} \quad \end{array} \right.
\]

then

\[
a_{1} e^{\frac{-K(t+(n+i)\nu)}{T_i}} \sin \frac{2\pi}{T_i} (t+\gamma_1 (n+i)\nu)
\]

\[
=a_{1} (1-K_{1} i \nu) e^{\frac{-K_{1}(t+i \nu)}{T_i}} \sin \frac{2\pi}{T_i} (t+\gamma_1 +(n+i)\nu)
\]

and

\[
a_{1} e^{\frac{-K(t+(n-i)\nu)}{T_i}} \sin \frac{2\pi}{T_i} (t+\gamma_1 +(n-i)\nu)
\]

\[
=a_{1} (1+K_{1} i \nu) e^{\frac{-K_{1}(t-i \nu)}{T_i}} \sin \frac{2\pi}{T_i} (t+\gamma_1 +(n-i)\nu)
\]

\[\text{—[40]—}\]
On a Period Analysis for Musical sound of Certain Musical Instrument of Percussion By R. Hiraga.

\[
\begin{align*}
\sum_{s=0}^{2n} a_1 e^{-K_1(t+np\nu)} \sin \frac{2\pi}{T_1} (t+\gamma_1 + \alpha \nu) \\
= a_1 e^{-K_1(t+np\nu)} \sin \frac{2\pi}{T_1} (t+\gamma_1 + n\nu) \left( \frac{\sin \left( \frac{(2n+1)\pi\nu}{T_1} \right)}{\sin \frac{\pi\nu}{T_1}} \right) \\
-2K_1\nu a_1 e^{-K_1(t+np\nu)} \cos \frac{2\pi}{T_1} (t+\gamma_1 + n\nu) \sum_{s=1}^{n} N \sin \frac{2N\pi}{T_1} \nu
\end{align*}
\]

consequently we have

\[
\frac{1}{(2n+1)^{2\nu}} \int_0^1 \left\{ \sum_{s=0}^{2n} a_1 e^{-K_1(t+np\nu)} \sin \frac{2\pi}{T_1} (t+\gamma_1 + \alpha \nu) \right\}^2 dt
\]

\[
= \frac{a_1^2 e^{-2K_1\nu}}{2(2n+1)^{2\nu}} \left( \frac{1 - e^{-2K_1\nu}}{2K_1} \right) \left( \frac{\sin \left( \frac{(2n+1)\pi\nu}{T_1} \right)}{\sin \frac{\pi\nu}{T_1}} \right)^2 + 4K_1^2\nu^2 \left( \sum_{s=1}^{n} N \sin \frac{2N\pi\nu}{T_1} \right)^2
\]

\[
+ 4K_1^2\nu^2 \left( \sum_{s=1}^{n} N \sin \frac{2N\pi\nu}{T_1} \right)^2 \left( \frac{e^{-2K_1\nu}}{4\pi} \sin \frac{4\pi}{T_1} (t+\gamma_1 + n\nu + \frac{T_1\delta}{4\pi}) \right)^\nu
\]

\[
- \frac{a_1^2 e^{-2K_1\nu}}{(2n+1)^{2\nu}} K_1^2\nu^2 \left( \frac{\sin \left( \frac{(2n+1)\pi\nu}{T_1} \right)}{\sin \frac{\pi\nu}{T_1}} \right)^\nu \sum_{s=1}^{n} N \sin \frac{2N\pi\nu}{T_1} \nu
\]

\[
\times \left( \frac{e^{-2K_1\nu}}{4\pi} \cos \frac{4\pi}{T_1} (t+\gamma_1 + n\nu + \frac{T_1\delta}{4\pi}) \right)^\nu
\]

\[
\frac{4\pi}{T_1} \sqrt{1 + \frac{K_1^2 T_1^2}{4\pi^2}}
\]

where \( \tan \delta = -\frac{K_1 T_1}{2\pi} \).

But

\[
\left| \frac{e^{-2K_1\nu} \sin \frac{4\pi}{T_1} (t+\gamma_1 + n\nu + \frac{T_1\delta}{4\pi}) \nu}{4\pi} \right| \frac{1}{T_1} \sqrt{1 + \frac{K_1^2 T_1^2}{4\pi^2}} < 2\pi \left( \frac{1}{T_1} \sqrt{1 + \frac{K_1^2 T_1^2}{4\pi^2}} \right)
\]

\[
\sim \{ 41 \}
\]
In our experiment, we have \( \frac{1}{T_1} > 100 \)

So that

\[
\left| \frac{e^{-2K_1\pi T_1^2}(t+\gamma_1+n\nu_1+T_1\delta)}{4\pi T_1^2 \sqrt{1+K_1^2 T_1^2}} \right|_0^\infty < \frac{1}{600 \sqrt{1+K_1^2 T_1^2}} < \frac{1}{600}
\]

Similarly

\[
\left| \frac{e^{-2K_1\pi T_1^2}(t+\gamma_1+n\nu_1+T_1\delta)}{4\pi T_1^2 \sqrt{1+K_1^2 T_1^2}} \right|_0^\infty < \frac{1}{600}
\]

Again

\[
\left( \frac{\sin((2n+1)\pi \nu_1 T_1)}{\sin \frac{\pi \nu_1 T_1}{T_1}} \right) \leq 2n + 1
\]

and

\[
\sum_{n=1}^{\infty} N \sin \frac{2N\pi \nu_1}{T_1} < \frac{n(n+1)}{2}
\]

Now

\[
e^{-K_1(n+1)\nu_1} = 1 - K_1(n+1)\nu_1 > 0
\]

\[
\therefore \quad K_1\nu_1 < \frac{1}{n+1}
\]

Consequently, in calculation of (7) we can neglect terms which contains

\[
\left| \frac{e^{-2K_1\pi T_1^2}(t+\gamma_1+n\nu_1+T_1\delta)}{4\pi T_1^2 \sqrt{1+K_1^2 T_1^2}} \right|_0^\infty \quad \text{or} \quad \left| \frac{e^{-2K_1\pi T_1^2}(t+\gamma_1+n\nu_1+T_1\delta)}{4\pi T_1^2 \sqrt{1+K_1^2 T_1^2}} \right|_0^\infty
\]

in each value of \( p \).

Therefore (7) can be expressed followingly,

\[
\frac{1}{(2n+1)^2 \nu_1} \left\{ \sum_{a=0}^{2n} a_1 e^{-K_1(\gamma_1+n\nu_1)} \sin \frac{2\pi}{T_1}(t+\gamma_1+\alpha \nu_1) \right\}^2 dt
\]

\[
\quad \text{(42)}
\]
On a Period Analysis for Musical sound of Certain Musical Instrument of Percussion By R. Hiraga.

\[
\frac{a_k e^{-K_{npv}}}{2(2n+1)^2} \left( 1 - e^{-2K_{npv}} \right) \left( \frac{\text{Sin}(2n+1)\pi p \nu}{T_1} \right)^2 + 4K_1^2 p^2 \nu^2 (\sum \text{Sin} \frac{2N\pi p \nu}{T_1})^2
\]

\[= \sum_{m=0}^{n} \left( \frac{a_k e^{-K_{npv}}}{2(2n+1)^2} \left( 1 - e^{-2K_{npv}} \right) \left( \frac{\text{Sin}(2n+1)\pi p \nu}{T_1} \right)^2 + 4K_1^2 p^2 \nu^2 (\sum \text{Sin} \frac{2N\pi p \nu}{T_1})^2 \right) + \frac{\sigma^2_{ai}}{2n+1}
\]

where \( \sum_{m} \) is standard deviation of \( M \)'s

But

\[4K_1^2 p^2 \nu^2 (\sum \text{Sin} \frac{2N\pi p \nu}{T_1})^2 < 1 \quad \text{for} \quad p \nu = T_1,
\]

\[= 2n+1 \quad \text{for} \quad p \nu = T_1.
\]

and

\[
\left( \frac{\text{Sin}(2n+1)\pi p \nu}{T_1} \right)^2 = \text{fairly Small value} \quad \text{for} \quad p \nu = T_1,
\]

\[= (2n+1)^2 \quad \text{for} \quad p \nu = T_1.
\]

Therefore

\[
\text{max. Value of} \sum_{m=0}^{n} = \frac{a_k e^{-K_{npv}}}{2} + \frac{\sigma^2_{ai}}{2n+1} \quad \text{for} \quad p \nu = T_1.
\]

More over the standard deviation of \( u \)'s is given by following:

\[
\sigma^2 = \frac{1}{(2n+1) \nu} \int_{0}^{(2n+1) \nu} \left( a_k e^{-K_{npv}} \text{Sin} \frac{2\pi}{T_1} (t+\gamma_1) + \left( \sum a_k e^{-K_{npv}} \text{Sin} \frac{2\pi}{T_1} (t+\gamma_1) \right) \right)^2 dt
\]

\[= \frac{1}{(2n+1) \nu} \int_{0}^{(2n+1) \nu} a_k e^{-K_{npv}} \left( \text{Sin} \frac{2\pi}{T_1} (t+\gamma_1) \right)^2 dt + \frac{\sigma^2_{ai}}{2n+1}
\]

\[= \frac{a_k^2}{2(2n+1) \nu} \left( 1 - e^{-2K_{npv}} \right) + \frac{\sigma^2_{ai}}{2n+1}
\]

\[
\sigma^2 = \frac{a_k^2}{2(2n+1) \nu} \left( 1 - e^{-2K_{npv}} \right) + \frac{\sigma^2_{ai}}{2n+1}
\]
On a Period Analysis for Music sound of Certain Musical Instrument of Percussion By R. Hiraga.

But

\[
1 - e \frac{-2K_1(2n+1)p
}{2K_1(2n+1)p}\ = \ e \frac{-2K_1n p
}{2K_1(2n+1)p} - \frac{1+e}{2K_1(2n+1)p}
\]

\[
= e \frac{-2K_1 n p
}{2K_1(2n+1)p} - \frac{2K_1 n p
}{2K_1(2n+1)p} + \frac{2K_1(n+1)p}{2K_1(2n+1)p} \]

\[
= e^{-2K_1 n p}
\]

So (11) becomes the next,

\[
\sigma^2 = \frac{a_1^2 e^{-2K_1 n p}}{2} + \sigma^2_{\text{al}}
\]

we call the next expression "Periodgram Equation,”

\[
\eta^2 = \frac{\sum_{n=1}^{\text{in}}}{\sigma^2}
\]

(13) becomes as follows:-

\[
\eta^2 = \frac{a_1^2 e^{-2K_1 n p}}{2(2n+1)^2} \left\{ \left( \frac{\text{Sin}(2n+1)\pi p
}{T_1} \right)^2 + \frac{4K_1^2 \rho^2 \nu^2 \left( \sum N \text{Sin} \frac{2N \pi p
}{T_1} \right)^2}{2n+1} \right\} + \frac{\sigma^2_{\text{al}}}{2} + \frac{\sigma^2}{2}
\]

(14)

and we get the maximum value of \( \eta^2 \) for a wave whose amplitude is \( a_1 \), as follows:-

\[
\eta^2_{\text{max}} = \frac{a_1^2 e^{-2K_1 n p}}{2} + \frac{\sigma^2_{\text{al}}}{2n+1} + \frac{\sigma^2}{2} \quad \text{for } p = T_1.
\]

(15)

Expressions (14) and (15) be applicable for any component wave so if we draw a graph by using abscissa \( \rho \), and ordinate \( \eta^2 \), its peak give us point to be researched for a wave component of period \( T_1 \).

And amplitude \( a_1 \) is studied by the following treatments

\[
a_1 e^{-K_1 n p} = \sqrt{\frac{2n+1}{n} \left( \frac{\sum_{n=1}^{\text{in}} - \frac{\sigma^2_{\text{al}}}{2n+1}}{2n+1} \right)}
\]

\[
= \left( 44 \right)
\]
On a Period Analysis for Musical sound of Certain Musical Instrument of Percussion By R. Hiraga.

\[
\begin{align*}
\Sigma_{\tau_{1} m} &= \text{Standard deviation of } M'_{s} \text{ for } \rho_{v} = T_{1}, \\
\sigma_{\tau_{1}} &= \text{Standard deviation of } u's \text{ for } \rho_{v} = T_{1},
\end{align*}
\]

In (16) if we get \( \Sigma_{\tau_{1} m} \leq \frac{\sigma_{\tau_{1}}^2}{2n+1} \leq 0 \), \( a_{1} \) is no existence.

We do the same treatment for the arrangement of \( u's \) from the second line to the (2n+2) th line, and we have

\[
\begin{align*}
a_{1} e^{-K_{1} \left( a+1 \right) \tau_{1}} &= \sqrt{\frac{2n+1}{n}} \left( \Sigma_{\tau_{1} m} - \frac{\sigma_{\tau_{1} m}^2}{2n+1} \right) \\
&= \sqrt{\frac{2n+1}{n}} \left( \Sigma_{\tau_{1} m} - \frac{\sigma_{\tau_{1} m}^2}{2n+1} \right) \left[ 1 + \left\{ \frac{n}{2} \left( \log \left( \Sigma_{\tau_{1} m} - \frac{\sigma_{\tau_{1} m}^2}{2n+1} \right) - \log \left( \Sigma_{\tau_{1} m} - \frac{\sigma_{\tau_{1} m}^2}{2n+1} \right) \right) \right\} \right]
\end{align*}
\]

and

\[
\begin{align*}
a_{1} &= a_{1} e^{-K_{1} \tau_{1}} = e^{\tau_{1}}
\end{align*}
\]

From (16) and (17), logarithmic decrement and amplitude \( a_{1} \) are given as follows:

\[
\begin{align*}
-K_{1} T_{1} &= \frac{1}{4} \left\{ \log \left( \Sigma_{\tau_{1} m} - \frac{\sigma_{\tau_{1} m}^2}{2n+1} \right) - \log \left( \Sigma_{\tau_{1} m} - \frac{\sigma_{\tau_{1} m}^2}{2n+1} \right) \right\} \\
&= \sqrt{\frac{2n+1}{n}} \left( \Sigma_{\tau_{1} m} - \frac{\sigma_{\tau_{1} m}^2}{2n+1} \right) \left[ 1 + \left\{ \frac{n}{2} \left( \log \left( \Sigma_{\tau_{1} m} - \frac{\sigma_{\tau_{1} m}^2}{2n+1} \right) - \log \left( \Sigma_{\tau_{1} m} - \frac{\sigma_{\tau_{1} m}^2}{2n+1} \right) \right) \right\} \right]
\end{align*}
\]

Case (11)

When \( C \) is certain integer, we will consider the case in which a sound wave contains a wave component whose period \( CT_{1} \), that is, \( CT_{1} \) is \( C \) times the period \( T_{1} \).

In this case we have

\[
\begin{align*}
U_{i} &= \sum_{a=0}^{2n} \left( a_{1} e^{-K_{1} \left( a+\alpha \rho_{v} \right) \theta_{1}} \right) \sin \frac{2\pi}{T_{1}} \left( t + \gamma_{1} + \alpha \rho_{v} \right) \\
&+ \sum_{a=0}^{2n} \alpha_{1} e^{-K_{1} \left( a+\alpha \rho_{v} \right) \theta_{1}} \sin \frac{2\pi}{T_{1}} \left( t + \rho_{v} \right) \\
M_{2}^2 &= \frac{1}{(2n+1)^2} \left[ \sum_{a=0}^{2n} a_{1} e^{-K_{1} \left( a+\alpha \rho_{v} \right) \theta_{1}} \sin \frac{2\pi}{T_{1}} \left( t + \gamma_{1} + \alpha \rho_{v} \right) \right]^2
\end{align*}
\]

- [45] -
On a Period Analysis for Musical sound of Certain Musical Instrument of Percussion By R. Hiraga.

\[ + \frac{1}{(2n+1)^2} \left\{ \sum_{\nu=0}^{2n} a_{\nu e}^{-k_1 \nu} \sin \frac{2\pi}{C T_1} (t + \gamma_1 + \alpha \nu) \right\}^2 \]

\[ + \frac{1}{(2n+1)^2} \left\{ \sum_{\nu=0}^{m} \sum_{\nu=1}^{n=3} a_{\nu e}^{-k_1 \nu} \sin \frac{2\pi}{T_1} (t + \gamma_1 + \alpha \nu) \right\}^2 \]

\[ + a_{1e}^{-k_1} \left( \frac{\sin(2n+1)\pi \nu}{T_1} \right) \left( \frac{\sin(2n+1)\pi \nu}{CT_1} \right) \times \]

\[ \cos \frac{2\pi}{T_1} \left\{ \left( 1 + \frac{1}{C} \right) t + (\gamma_1 + \frac{\gamma_1}{C}) \right\} - C \cos \frac{2\pi}{T_1} \left\{ \left( 1 + \frac{1}{C} \right) t + (\gamma_1 + \frac{\gamma_1}{C}) \right\} \]

\[ + 2K_1 K_1 \rho \nu \left( \sum N \sin \frac{2N_{\rho \nu}}{T_1} \right) \left( \sum N \sin \frac{2N_{\rho \nu}}{CT_1} \right) \left( \cos \frac{2\pi}{T_1} \left\{ \left( 1 + \frac{1}{C} \right) t + (\gamma_1 + \frac{\gamma_1}{C}) \right\} \right) \]

\[ + \cos \frac{2\pi}{T_1} \left\{ \left( 1 + \frac{1}{C} \right) t + (\gamma_1 + \frac{\gamma_1}{C}) \right\} \]

\[ - k_1 \rho \nu \left( \frac{\sin(2n+1)\pi \nu}{CT_1} \right) \left( \sum N \sin \frac{2N_{\rho \nu}}{CT_1} \right) \left( \sin \frac{2\pi}{T_1} \left\{ \left( 1 + \frac{1}{C} \right) t + (\gamma_1 + \frac{\gamma_1}{C}) \right\} \right) \]

\[ + \sin \frac{2\pi}{T_1} \left\{ \left( 1 + \frac{1}{C} \right) t + (\gamma_1 + \frac{\gamma_1}{C}) \right\} \]

\[ - k_1 \rho \nu \left( \frac{\sin(2n+1)\pi \nu}{CT_1} \right) \left( \sum N \sin \frac{2N_{\rho \nu}}{CT_1} \right) \times \]

\[ \left[ \sin \frac{2\pi}{T_1} \left\{ \left( 1 + \frac{1}{C} \right) t + (\gamma_1 + \frac{\gamma_1}{C}) \right\} - \sin \frac{2\pi}{T_1} \left\{ \left( 1 + \frac{1}{C} \right) t + (\gamma_1 + \frac{\gamma_1}{C}) \right\} \right] \] (21)

\[ \frac{1}{\rho \nu} M_d \int dt = \frac{a_{1e}^{-2k_1 \nu}}{2(2n+1)^2} \left[ \left( \frac{\sin(2n+1)\pi \nu}{T_1} \right) \right] ^2 + 4k_1^2 \rho^2 \nu^2 \left( \sum N \sin \frac{2N_{\rho \nu}}{T_1} \right)^2 \]

\[ + a_{1e}^{-2k_1 \nu} \left( \frac{\sin(2n+1)\pi \nu}{CT_1} \right) \left( \sum N \sin \frac{2N_{\rho \nu}}{CT_1} \right)^2 \]

\[ + \frac{\sigma_{al}^2}{2n+1} \] (22)

And the standard deviation of u's is

\([-46\) --}
On a Period Analysis for Musical sound of Certain Musical Instrument of Percussion By R. Hiraga.

\[
\sigma^2 = \frac{1}{(2n+1)\rho^2} \int_0^{(2n+1)\rho^2} a_1^2 e^{-2K_1 t} \sin^2 \frac{2\pi}{T_1} (t + \gamma_1) + a_1^2 e^{-2K_{1'}} \sin^2 \frac{2\pi}{CT_1} (t + \gamma_{1'}) \\
+ a_1 a_1' e^{-K_1 + K_{1'}} (t + (\gamma_1 + \gamma_{1'})) \right) \cos \frac{2\pi}{T_1} \left( (1 - \frac{1}{C})t + (\gamma_1 + \gamma_{1'}) \right) \right] dt \\
\approx a_1^2 e^{-2K_1 \rho^2} + a_1^2 e^{-2K_{1'} \rho^2} + \sigma_{al}^2 \\
(23)
\]

So that the periodogram's equation is given by:

\[
\eta^2 = \frac{a_1^2 e^{-2K_1 \rho^2} \left\{ \left( - \frac{\sin(2n+1)\pi\rho^2}{CT_1} \right)^2 + 4K_1^2 \rho^2 \left( \sum_{n=0}^{\infty} \sin^2 \frac{2\pi n}{CT_1} \right) \right\}}{a_1^2 e^{-2K_1 \rho^2} + a_1^2 e^{-2K_{1'} \rho^2} + \sigma_{al}^2} + \sigma_{al}^2 \\
(24)
\]

The Maximum value of \( \eta^2 \) is given as the next:

\[
\eta_{max}^2 = \frac{\sum_{m=\infty}^{\infty} a_m^2 e^{-2K_1 \rho^2} + a_1^2 e^{-2K_{1'} \rho^2} = \sigma_{al}^2}{\sigma_{CT}^2} = \frac{\sum_{m=\infty}^{\infty} a_m^2 e^{-2K_1 \rho^2} + a_1^2 e^{-2K_{1'} \rho^2} + \sigma_{al}^2}{\sigma_{CT}^2} \\
(25)
\]

where \( \sum_{m=\infty}^{\infty} a_m^2 e^{-2K_1 \rho^2} = \text{standard deviation of } M's \text{ for } \rho \text{ is } CT_1 \)

\( \sigma_{CT}^2 = \text{standard deviation of } u's \text{ for } \rho = CT_1 \)

Now we adopt the central term which is used in calculation of \( \sum_{m=\infty}^{\infty} a_m^2 e^{-2K_1 \rho^2} \) and \( \sigma_{CT}^2 \) of (25) as the central term, and we try the same method as the expression (16).

Then we have

\[
a_1^2 e^{-2K_1 \rho^2} \equiv A
\]

So that

\[
a_1^2 e^{-K_{1'} \rho^2} = \sqrt{\frac{2n+1}{n} \sum_{m=\infty}^{\infty} a_m^2 e^{-2K_1 \rho^2} - \sigma_{al}^2} - A
\]

\(-47°-\)
But $a'_1$ is no existence for

$$\frac{2n+1}{n} \sum \frac{\sigma^2_{CT_1}}{n} - A \leq 0$$

Next we try the above method by using $u$'s from the second line to the $(2n+2)$ th line, and we have

$$a'_1 e^{-K_j^{(n+1)CT_1}} = \sqrt{\frac{2n+1}{n} \sum \frac{\sigma^2_{CT_1}}{n} - \frac{\sigma^2_{CT_1}}{n} - A'}$$

(27)

where

$$\Sigma_{u(CT_1)} = \text{standard deviation of } M_s \text{ for } p = CT_1,$$

$$\sigma_{CT_1} = \text{standard deviation of } u_s \text{ for } p = CT_1,$$

$$A'_1 = e^{2} e^{-2K_j^{(n+1)CT_1}}$$

From (26) and (27), logarithmic decrement is given as follows,

$$-\frac{CT_1 K_j'}{2} = \frac{1}{4} \left\{ \log \left( \frac{2n+1}{n} \sum \frac{\sigma^2_{CT_1}}{n} - \frac{\sigma^2_{CT_1}}{n} - A' \right) - \log \left( \frac{2n+1}{n} \sum \frac{\sigma^2_{CT_1}}{n} - \frac{\sigma^2_{CT_1}}{n} - A \right) \right\}$$

(28)

Again amplitude $a'_1$ is given as follows,

$$a'_1 = \sqrt{\frac{2n+1}{n} \sum \frac{\sigma^2_{CT_1}}{n} - \frac{\sigma^2_{CT_1}}{n} - A}$$

$$\times \left[ 1 + \frac{n}{2} \left\{ \log \left( \frac{2n+1}{n} \sum \frac{\sigma^2_{CT_1}}{n} - \frac{\sigma^2_{CT_1}}{n} - A' \right) - \log \left( \frac{2n+1}{n} \sum \frac{\sigma^2_{CT_1}}{n} - \frac{\sigma^2_{CT_1}}{n} - A \right) \right\} \right]$$

(29)

When there is a component wave with a period $bT_1$ which has an integral multiple period of $CT_1$, we can get the result as the same method as the method of (26), (27), (28) and (29).

Conclusion

From the above reasoning we can conclude shortly:

(1) To find the periods of component waves is by drawing the graph of Periodogram Equation. Of course this method is not necessary, but paying
On a Period Analysis for Musical sound of Certain Musical Instrument of Percussion By R. Hiraga.

attention to the maximum values of \( \eta^2 \) is sufficient.

(2) Existence of component wave is decided by calculation.

(3) Logarithmic decrements and amplitudes of component waves are had by calculation.

**Application**

**Material:**–

An oscillogram of a bell \( \ddagger k \) which is one of Korean Shrine Festival musical instruments at Confucius Shrine in Seoul, and we used the properly enlarged parts of the oscillogram which is taken by Electro-magnetic Oscillograph.

**Example (A)**

After striking, \[ \frac{0.3}{100} \sim \frac{4.3}{100} \] sec.

Enlarged rate, time interval \[ \frac{1}{100} \] sec. to 93 cm.

We get table (I) and (II) from the above material.

**Table (I) sound wave material of Bell \( \ddagger k \)**

| 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 1  | 43 | 46.0| 50 | 65 | 45 | 27 | 29 | 60 | 57 | 45 | 40 | 36 | 65 | 54 | 55 | 54 | 40 | 40 | 60 | 75 | 50 | 45 | 50 | 32 | 35 | 40 | 34 | 30 | 25 | 20 | 15 | 10 |
| 2  | 60 | 55.0| 45 | 30 | 60 | 40 | 27 | 55 | 55 | 45 | 40 | 36 | 65 | 54 | 55 | 54 | 35 | 40 | 40 | 60 | 75 | 50 | 45 | 50 | 32 | 35 | 40 | 34 | 30 | 25 | 20 | 15 | 10 |
| 3  | 62 | 25.0| 40 | 30 | 60 | 40 | 27 | 55 | 55 | 45 | 40 | 36 | 65 | 54 | 55 | 54 | 35 | 40 | 40 | 60 | 75 | 50 | 45 | 50 | 32 | 35 | 40 | 34 | 30 | 25 | 20 | 15 | 10 |
| 4  | 63 | 34.0| 40 | 30 | 60 | 40 | 27 | 55 | 55 | 45 | 40 | 36 | 65 | 54 | 55 | 54 | 35 | 40 | 40 | 60 | 75 | 50 | 45 | 50 | 32 | 35 | 40 | 34 | 30 | 25 | 20 | 15 | 10 |
| 5  | 62 | 34.0| 40 | 30 | 60 | 40 | 27 | 55 | 55 | 45 | 40 | 36 | 65 | 54 | 55 | 54 | 35 | 40 | 40 | 60 | 75 | 50 | 45 | 50 | 32 | 35 | 40 | 34 | 30 | 25 | 20 | 15 | 10 |
| 6  | 70 | 33.0| 30 | 60 | 40 | 27 | 55 | 55 | 45 | 40 | 36 | 65 | 54 | 55 | 54 | 35 | 40 | 40 | 60 | 75 | 50 | 45 | 50 | 32 | 35 | 40 | 34 | 30 | 25 | 20 | 15 | 10 |
| 7  | 70 | 33.0| 30 | 60 | 40 | 27 | 55 | 55 | 45 | 40 | 36 | 65 | 54 | 55 | 54 | 35 | 40 | 40 | 60 | 75 | 50 | 45 | 50 | 32 | 35 | 40 | 34 | 30 | 25 | 20 | 15 | 10 |
| 8  | 70 | 33.0| 30 | 60 | 40 | 27 | 55 | 55 | 45 | 40 | 36 | 65 | 54 | 55 | 54 | 35 | 40 | 40 | 60 | 75 | 50 | 45 | 50 | 32 | 35 | 40 | 34 | 30 | 25 | 20 | 15 | 10 |
| 9  | 70 | 33.0| 30 | 60 | 40 | 27 | 55 | 55 | 45 | 40 | 36 | 65 | 54 | 55 | 54 | 35 | 40 | 40 | 60 | 75 | 50 | 45 | 50 | 32 | 35 | 40 | 34 | 30 | 25 | 20 | 15 | 10 |
| 10 | 70 | 33.0| 30 | 60 | 40 | 27 | 55 | 55 | 45 | 40 | 36 | 65 | 54 | 55 | 54 | 35 | 40 | 40 | 60 | 75 | 50 | 45 | 50 | 32 | 35 | 40 | 34 | 30 | 25 | 20 | 15 | 10 |
| 11 | 70 | 33.0| 30 | 60 | 40 | 27 | 55 | 55 | 45 | 40 | 36 | 65 | 54 | 55 | 54 | 35 | 40 | 40 | 60 | 75 | 50 | 45 | 50 | 32 | 35 | 40 | 34 | 30 | 25 | 20 | 15 | 10 |
| 12 | 70 | 33.0| 30 | 60 | 40 | 27 | 55 | 55 | 45 | 40 | 36 | 65 | 54 | 55 | 54 | 35 | 40 | 40 | 60 | 75 | 50 | 45 | 50 | 32 | 35 | 40 | 34 | 30 | 25 | 20 | 15 | 10 |
| 13 | 70 | 33.0| 30 | 60 | 40 | 27 | 55 | 55 | 45 | 40 | 36 | 65 | 54 | 55 | 54 | 35 | 40 | 40 | 60 | 75 | 50 | 45 | 50 | 32 | 35 | 40 | 34 | 30 | 25 | 20 | 15 | 10 |
| 14 | 70 | 33.0| 30 | 60 | 40 | 27 | 55 | 55 | 45 | 40 | 36 | 65 | 54 | 55 | 54 | 35 | 40 | 40 | 60 | 75 | 50 | 45 | 50 | 32 | 35 | 40 | 34 | 30 | 25 | 20 | 15 | 10 |
| 15 | 70 | 33.0| 30 | 60 | 40 | 27 | 55 | 55 | 45 | 40 | 36 | 65 | 54 | 55 | 54 | 35 | 40 | 40 | 60 | 75 | 50 | 45 | 50 | 32 | 35 | 40 | 34 | 30 | 25 | 20 | 15 | 10 |
| 16 | 70 | 33.0| 30 | 60 | 40 | 27 | 55 | 55 | 45 | 40 | 36 | 65 | 54 | 55 | 54 | 35 | 40 | 40 | 60 | 75 | 50 | 45 | 50 | 32 | 35 | 40 | 34 | 30 | 25 | 20 | 15 | 10 |
On a Period Analysis for Musical sound of Certain Musical Instrument of Percussion By R. Hiraga.

Table (II) Table of \( \eta^2 = \frac{\Sigma^2}{\sigma^2} \)

<table>
<thead>
<tr>
<th>( \mu^2 )</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.005</td>
<td>0.005</td>
<td>0.039</td>
<td>0.015</td>
<td>0.219</td>
<td>0.009</td>
<td>0.014</td>
<td>0.041</td>
<td>0.023</td>
<td>0.083</td>
<td>0.064</td>
<td>0.046</td>
<td>0.021</td>
</tr>
</tbody>
</table>

| \( \Sigma^2 \) | 0.038 | 0.395 | 0.033 | 0.029 | 0.027 | 0.082 | 0.128 | 0.044 | 0.042 | 0.024 |

From the graph (fig. 1) by the second table, we have six researching needful points of (4), (6), (9), (11), (16) and (22).

At (4) \( n=7, \quad \frac{15}{7} \Sigma^2 - \frac{\sigma^2}{7} = -12.10 < 0\)

\( \therefore a_n \) is no existence.

At (6) \( \frac{15}{7} \Sigma^2 - \frac{\sigma^2}{7} = 62.41\)

\( \frac{15}{7} \Sigma^2 - \frac{\sigma^2}{7} = 47.36\)

\( \frac{TK}{2} = -0.069\)

\( a_n = 15.52\)

frequency = 1550/sec.
On a Period Analysis for Musical Sound of Certain Musical Instrument of Percussion By R. Hiraga.

At (9) \[ \frac{15}{7} \Sigma^2 - \frac{\sigma^2}{7} = -11.84 < 0 \]

\( a_8 \) is no existence.

At (11) \[ \frac{15}{7} \Sigma^2 - \frac{\sigma^2}{7} = 8.60 \]

\[ \frac{15}{7} \Sigma^2 - \frac{\sigma'^2}{7} = 1.70 \]

\[ -\frac{TK}{2} = -0.176 \]

\( a_{11} = 19.54 \)

frequency = 846/sec.

the amplitude decreases rapidly.

At (22) \[ \frac{15}{7} \Sigma^2 - \frac{\sigma^2}{7} = 49.31 \]

\[ A = \frac{31}{15} \Sigma^2 - \frac{\sigma^2}{15} = 0.02 \]

\[ \frac{15}{7} \Sigma'^2 - \frac{\sigma'^2}{7} = 57.38 \]

\[ A' = \frac{31}{15} \Sigma'^2 - \frac{\sigma'^2}{15} = 0.01 \]

So \[ \frac{15}{7} \Sigma^2 - \frac{\sigma^2}{7} - A = 49.29 \]

\[ \frac{15}{7} \Sigma'^2 - \frac{\sigma'^2}{7} - A' = 57.37 \]

\[ -\frac{TK}{2} = 0.0378 \]

frequency = 423/sec.

\( a_{23} = 2.59 \)

\[ \therefore \frac{a_8}{79} = \frac{a_{11}}{100} = \frac{a_{16}}{25} = \frac{a_{23}}{18} \]

Example (B)

After striking, \( \left( -\frac{19}{100} \sim -\frac{23.1}{100} \right) \) sec.

–[51]–
Enlarged rate, time interval \( \frac{1}{100} \) sec. to 99.5 cm.

We have table (III) and (IV) from the above material.

**Table (III) sound wave material of Bell**

|   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 1 | 29.0 31.0 54.0 61.0 54.0 36.0 35.0 43.0 54.6 61.0 49.3 31.0 40.0 43.0 54.5 49.0 37.0 26.3 32.0 45.0 54.0 60.0 41.0 31.0 |
| 2 | 36.0 55.0 64.0 49.0 33.0 29.0 53.0 32.0 37.0 51.0 58.0 37.0 26.8 28.0 36.0 66.0 66.0 50.0 26.0 32.0 42.0 60.0 59.0 40.0 29.5 5.0 |
| 3 | 29.0 56.0 61.0 54.0 32.0 24.0 0.0 45.0 64.0 70.0 55.0 28.0 30.0 40.0 56.0 58.0 50.0 30.0 23.0 52.5 55.0 60.0 51.0 0.3 21.0 40.0 |
| 4 | 67.0 65.0 43.0 36.0 35.0 54.0 0.0 47.0 52.0 40.0 32.0 29.0 31.5 55.0 58.0 43.0 30.0 31.0 50.0 59.0 50.0 43.0 36.0 39.0 47.0 0.0 |
| 5 | 42.5 37.0 26.0 25.45.0 58.0 0.0 45.0 35.0 33.45.0 60.0 52.39.0 42.0 41.0 42.46.5 48.0 40.0 28.5 23.0 40.0 50.0 56.0 14.5 0.0 |
| 6 | 32.0 31.0 60.0 65.45.0 38.0 0.0 37.0 38.46.55.0 51.0 38.26.0 26.0 54.0 66.45.0 32.0 27.0 37.0 55.0 63.0 56.0 45.35.0 0.0 |
| 7 | 29.0 35.0 68.0 60.43.0 21.0 20.0 30.0 59.68.0 60.0 35.27.0 40.0 56.0 59.55.0 45.0 34.25.0 56.0 61.0 57.0 40.19.0 0.0 |
| 8 | 28.0 67.0 60.44.36.0 56.0 43.0 57.0 59.50.0 37.0 31.35.0 58.0 60.0 40.0 30.0 26.0 50.56.0 49.0 43.5 36.38.55.0 0.0 |
| 9 | 54.0 49.0 5.46.33.31.0 50.0 55.0 35.32.95.0 45.55.45.0 54.0 40.0 41.46.0 54.0 53.45.5 37.0 28.0 50.0 59.43.0 0.0 |
| 10 | 31.0 26.0 50.0 61.54.0 0.0 40.0 37.0 40.46.57.0 54.0 51.32.0 26.0 35.0 53.61.0 45.0 30.21.0 31.60.0 58.45.36.0 0.0 |
| 11 | 31.0 40.0 60.0 67.59.0 43.0 26.0 30.57.62.0 54.0 34.22.0 42.0 57.0 61.49.0 39.0 29.41.0 65.65.0 49.0 30.27.0 0.0 |
| 12 | 50.0 54.0 5.55.34.29.5 45.0 52.0 59.57.44.0 33.44.52.0 64.0 51.0 28.30.0 45.0 52.47.0 39.0 31.0 35.50.59.0 0.0 |
| 13 | 53.0 42.0 38.40.50.0 58.0 50.0 37.33.40.0 6.45.49.49.0 37.0 32.54.55.5 58.0 50.0 35.0 36.53.0 60.51.36.0 0.0 |
| 14 | 30.5 36.0 50.53.45.0 34.0 30.0 45.56.61.5 55.39.0 30.0 53.0 62.0 55.42.0 27.0 26.46.0 57.0 50.5 36.27.37.0 0.0 |
| 15 | 31.0 38.0 53.30.31.0 45.0 5.54.0 58.51.31.0 28.43.52.0 61.0 54.0 23.35.0 55.0 66.61.5 45.34.0 39.48.59.0 0.0 |
| 16 | 50.0 31.0 27.37.52.0 62.0 60.0 28.34.47.0 58.51.32.5 39.0 46.5 53.53.0 40.0 31.31.0 45.55.0 56.40.40.0 0.0 |

**Table (IV) Table of \( \eta^2 = \frac{\sum^2}{\sigma^2} \)**

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \eta^2 )</td>
<td>0.002</td>
<td>0.014</td>
<td>0.004</td>
<td>0.016</td>
<td>0.650</td>
<td>0.069</td>
<td>0.020</td>
<td>0.019</td>
<td>0.018</td>
<td>0.014</td>
<td>0.347</td>
<td>0.047</td>
<td>0.015</td>
</tr>
<tr>
<td>15</td>
<td>16</td>
<td>17</td>
<td>18</td>
<td>19</td>
<td>20</td>
<td>21</td>
<td>22</td>
<td>23</td>
<td>24</td>
<td>25</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \eta^2 )</td>
<td>0.011</td>
<td>0.030</td>
<td>0.064</td>
<td>0.220</td>
<td>0.084</td>
<td>0.076</td>
<td>0.009</td>
<td>0.033</td>
<td>0.113</td>
<td>0.210</td>
<td>0.118</td>
<td>0.099</td>
<td></td>
</tr>
</tbody>
</table>

From the graph (Fig. 2) by the fourth table, we have four researching needful pints of (6), (12), (18) and (23.5).

At (6) \( \frac{15}{7} \sum^2 - \frac{\sigma^2}{7} = 240.34 \)
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\[ \frac{15}{7} \sum_{i} \sigma_{i}^2 - \frac{\sigma_{1}^2}{7} = 247.30 \]

\[ \frac{KT}{2} = 0.00718 \]

\[ a_{6} = 15.11 \]

frequency = 1658/sec.

At (12)

\[ \frac{15}{7} \sum_{i} \sigma_{i}^2 - \frac{\sigma_{1}^2}{7} = 96.32 \]

\[ \frac{15}{7} \sum_{i} \sigma_{i}^2 - \frac{\sigma_{1}^2}{7} = 77.78 \]

\[ A = a_{6} e^{-K_{1} \sigma_{1}^2} = a_{6} e^{-2K_{1} \sigma_{1}^2} = \frac{31}{15} \sum_{i} \sigma_{i}^2 - \frac{\sigma_{1}^2}{15} = 171.36 \]

\[ A' = a_{6} e^{-K_{1} (n+1) \sigma_{1}^2} = a_{6} e^{-2K_{1} (n+1) \sigma_{1}^2} = \frac{31}{15} \sum_{i} \sigma_{i}^2 - \frac{\sigma_{1}^2}{15} = 165.48 \]

then

\[ \frac{15}{7} \sum_{i} \sigma_{i}^2 - \frac{\sigma_{1}^2}{7} - A = -75.04 < 0 \]

So that \( a_{12} \) is no existence. But by the research for the changing state of a component whose amplitude \( a_{6} \), We have

\(-[53]-\)
On a Period Analysis for Musical sound of Certain Musical Instrument of Percussion By R. Uyraga.

\[-\frac{KT}{2} = -0.0086\]

\[a_6 = 15.77\]

It shows, the component wave which has an amplitude \(a_6\) is in decreasing state through increasing state and it has a nearly equal amplitude in the time interval of 15 times of its own period.

At (18)

\[\frac{15}{7} \sum_2 - \frac{\sigma^2}{7} = 49.14\]

\[\frac{15}{7} \sum'2 - \frac{\sigma'^2}{7} = 47.77\]

\[A = \frac{47}{23} \sum_2 - \frac{\sigma^2}{23} = 36.70\]

\[A' = \frac{47}{23} \sum'_2 - \frac{\sigma'^2}{23} = 36.58\]

\[\therefore \frac{15}{7} \sum_2 - \frac{\sigma^2}{7} - A = 12.44\]

\[\frac{15}{7} \sum'2 - \frac{\sigma'^2}{7} - A' = 11.19\]

So we have

\[-\frac{KT}{2} = -0.02932\]

\[a_{18} = 4.71\]

frequency = 553/sec.

At (23.5)

\[\frac{15}{7} \sum_2 - \frac{\sigma^2}{7} = 1.54\]

\[\frac{15}{7} \sum'2 - \frac{\sigma'^2}{7} = 1.79\]

\[-\frac{KT}{2} = 0.0375\]

\[a_{23.5} = 0.588\]
On a Period Analysis for Musical sound of Certain Musical Instrument of Percussion By R. Hiraga.

frequency\(=424/\text{sec.}\)

and so

\[
\frac{a_5}{100} = \frac{a_{18}}{31} = \frac{a_{23.5}}{4}
\]

Example (C)

After striking \(\left( \frac{39}{100} \sim \frac{43}{100} \right)\) sec.

Enlarged rate, time interval \(\frac{1}{100}\) sec. to 99, 5 cm.

We have the table (V) and the (VI) from the above material.

Table (V) sound wave material of Bell \(\text{uk}\)

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>40.0</td>
<td>47.0</td>
<td>55.0</td>
<td>50.0</td>
<td>42.0</td>
<td>.42.5</td>
<td>.46.5</td>
<td>.53.0</td>
<td>.53.5</td>
<td>41.0</td>
<td>36.0</td>
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<tr>
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- (55-)
Ona Period Analysis for Musical sound of Certain Musical Instrument of Percussion by R. Hiraga.

| 52.0 | 39.0 | 33.0 | 36.0 | 47 | 55.5 | 41.0 | 35.0 | 32.5 | 35.0 | 42.0 | 53.0 |
| 37.0 | 32.0 | 36.0 | 43.0 | 50 | 49.0 | 43.0 | 38.0 | 36.5 | 52.0 | 56.0 | 52.0 |
| 37.5 | 44.0 | 52.0 | 52.0 | 46 | 42.0 | 40.0 | 45.0 | 47.5 | 44.0 | 39.5 | 40.0 |
| 50.0 | 47.5 | 41.0 | 40.0 | 37 | 39.0 | 43.0 | 39.0 | 45.0 | 40.0 | 41.0 | 47.0 |
| 50.5 | 45.5 | 36.0 | 36.0 | 52 | 58.0 | 51.0 | 42.0 | 37.0 | 35.0 | 45.0 | 50.0 |
| 40.0 | 35.0 | 37.0 | 45.0 | 50 | 48.0 | 40.0 | 34.0 | 33.5 | 45.0 | 53.0 | 55.0 |
| 35.5 | 37.0 | 46.0 | 51.5 | 49 | 45.0 | 42.2 | 43.5 | 47.0 | 50.5 | 48.5 | 44.0 |
| 41.5 | 46.0 | 51.5 | 48.0 | 44 | 43.0 | 40.0 | 39.0 | 45.0 | 45.0 | 45.0 | 42.5 |
| 34.0 | 40.0 | 49.0 | 43.0 | 46 | 37.0 | 35.0 | 45.0 | 54.0 | 56.0 | 47.0 | 41.0 |
| 35.0 | 45.0 | 53.5 | 64.0 | 47 | 39.0 | 38.0 | 40.5 | 48.0 | 53.0 | 49.0 | 39.0 |

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| 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 |
|---|
| 0.012 | 0.019 | 0.024 | 0.055 | 0.233 | 0.047 | 0.039 | 0.022 | 0.020 | 0.012 | 0.017 | 0.091 | 0.023 |

![Graph](fig.3)

From the graph (fig.3) which is described by the table (VI), we have four researching needful pints (6), (12), (18) and (25),—
At (6)

\[
\frac{15}{7} \Sigma^2 - \frac{\sigma^2}{7} = 66.09
\]

\[
\frac{15}{7} \Sigma'^2 - \frac{\sigma'^2}{7} = 74.36
\]

\[-\frac{KT}{2} = 0.02944\]

\[a_n = 7.293\]

frequency = 1658/sec.

The amplitude is in increasing state.

At (12)

\[
\frac{15}{7} \Sigma^2 - \frac{\sigma^2}{7} = 27.79
\]

\[
\frac{15}{7} \Sigma'^2 - \frac{\sigma'^2}{7} = 25.40
\]

\[A = 24.81\]

\[A' = 22.38\]

\[\therefore \frac{15}{7} \Sigma^2 - \frac{\sigma^2}{7} - A = 2.98\]

\[\frac{15}{7} \Sigma'^2 - \frac{\sigma'^2}{7} - A' = 3.02\]

\[-\frac{KT}{2} = 0.0038\]

\[a_{12} = 1.65\]

frequency = 829/sec.

At (18)

\[
\frac{15}{7} \Sigma^2 - \frac{\sigma^2}{7} = 3.44
\]

\[
\frac{15}{7} \Sigma'^2 - \frac{\sigma'^2}{7} = 3.14
\]

\[A \equiv \frac{47}{23} \Sigma_2^2 - \frac{\sigma_2^2}{23} = 3.00\]

\[-(57)\]
A' = \frac{47}{23} \sum \frac{a'^2}{2} - \frac{\sigma'^2}{23} = 2.77

then

\frac{15}{7} \sum \frac{a^2}{7} - \frac{\sigma^2}{7} - A = 0.44

\frac{15}{7} \sum \frac{a'^2}{7} - \frac{\sigma'^2}{7} - A' = 0.37

- \frac{KT}{2} = -0.043

a_{18} = 1.06

frequency = 553/sec.

At (25)

\frac{15}{7} \sum \frac{a^2}{7} = 1.94

\frac{15}{7} \sum \frac{a'^2}{7} = 2.13

- \frac{KT}{2} = 0.0231

a_{25} = 0.939

frequency = 398/sec.

So we have

\frac{a_6}{100} = \frac{a_{13}}{23} = \frac{a_{18}}{14} = \frac{a_{25}}{13}

The results of three example shown by acoustic spectrums of fig 4, and a used oscillogram of the bell of Korean Shrine Festival musical Instruments at Confucius Shrine in Seoul is shown by fig. 5.
fig 5. Oscillogram of a Sound of Bell
Ona Period Analysis for Musical sound of Certain Musical Instrument of Percussion by E. Hiraga.

B part

C part