FACTORIZATION OF POSITIVE CONES OF ORDER n OF VON NEUMANN ALGEBRAS

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Dedicated to Professor Jun Tomiyama on his 60th birthday

ABSTRACT. In this paper we shall consider the factorization of positive cones of order n of a von Neumann algebra. Namely, we shall show the existence of a $*$ -subalgebra inducing the positive cone of order n of a von Neumann algebra.

Let M be a von Neumann algebra on a Hilbert space H . It is known that a positive cone $M_n(M)^+ (= (M \otimes M_n)^+)$, where M_n denotes an algebra of all $n \times n$ matrices, coincides with the convex hull of all elements $[x_i^* x_j]$ for $x_i \in M$. We shall find a $*$ -subalgebra N of M such that a positive cone of order n is generated by all elements $[a_i^* c^* c a_j]$ for $a_i \in N$, $c \in M$. We may then say that the cone is factorized by N . The purpose of this paper is to consider when a positive cone of order n can be factorized or not. The factorization of a self-dual cone in the Hilbert space associated with a standard form of a von Neumann algebra was already shown in [4]. We shall use the books of Pedersen [5] and Takesaki [8] as references of concepts and results of operator algebras.

We begin with the following definition, in which we consider two convex cones of order n .

Definition 1. Let M be a von Neumann algebra on a Hilbert space H and N a subspace of M . For a natural number n we put

$$C_n(N) = \overline{\text{co}}^s \{ [a_i^* c^* c a_j] \in M_n(M) \mid a_i \in N, c \in M \},$$

$$D_n(N) = \left\{ [x_{i,j}] \in M_n(M) \mid \sum_{i,j=1}^n a_i^* x_{i,j} a_j \geq 0 \text{ for every } a_1, \dots, a_n \in N \right\},$$

where $\overline{\text{co}}^s$ denotes the strongly closed convex hull.

It is clear that $C_n(N) \subset M_n(M)^+ \subset D_n(N)$. In particular, if $N = M$, these cones are equal.

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We now show the condition for N that $D_n(N)$ coincides with the positive cone of order n of M .

Proposition 2. *Let M be a von Neumann algebra on a Hilbert space H and π a faithful $*$ -representation of M on a Hilbert space K . If N is a subspace of M such that $\pi(N)$ has a cyclic vector (i.e., $\pi(N)\xi$ is dense in K for some $\xi \in K$), then $M_n(M)^+ = D_n(N)$, $n \in \mathbb{N}$.*

Proof. Choose the element ξ in K as above. If $[x_{i,j}]$ belongs to $D_n(N)$, then

$$\begin{aligned} & \left(\begin{bmatrix} \pi(x_{1,1}) & \cdots & \pi(x_{1,n}) \\ \vdots & & \vdots \\ \pi(x_{n,1}) & \cdots & \pi(x_{n,n}) \end{bmatrix} \begin{bmatrix} \pi(a_1)\xi \\ \vdots \\ \pi(a_n)\xi \end{bmatrix}, \begin{bmatrix} \pi(a_1)\xi \\ \vdots \\ \pi(a_n)\xi \end{bmatrix} \right) \\ &= \left(\sum_{i,j=1}^n \pi(x_{i,j})\pi(a_j)\xi, \pi(a_i)\xi \right) \\ &= \left(\pi \left(\sum_{i,j=1}^n a_i^* x_{i,j} a_j \right) \xi, \xi \right) \geq 0, \end{aligned}$$

for every $a_i \in N$. Therefore, $[\pi(x_{i,j})] \geq 0$, so $[x_{i,j}] \geq 0$. It follows that $[x_{i,j}]$ belongs to $M_n(M)^+$. This completes the proof.

We immediately obtain the following examples from the above proposition.

Example 3. Suppose that

$$M = M_{k(1)} \oplus \cdots \oplus M_{k(m)}, \quad k(1) \leq \cdots \leq k(m).$$

Put

$$A = \left\{ \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots \\ 0 & & \lambda_{k(1)} \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots \\ 0 & & \lambda_{k(m)} \end{bmatrix} \mid \lambda_1, \lambda_2, \dots, \lambda_{k(m)} \in \mathbb{C} \right\}.$$

Since the subalgebra of all diagonal matrices of $M_{k(i)}$ has a cyclic vector in $H_{k(i)}$ where H_k means a k -dimensional Hilbert space, $M_n(M)^+$ coincides with $D_n(A)$ for $n \in \mathbb{N}$ by the above proposition (cf. [4, Lemma 2.1]).

Another example is given by $M_n(M_{km})^+ = D_n(M_k \otimes I_m)$, $n \in \mathbb{N}$, with $k \geq m$.

Lemma 4. *Let M be a σ -finite, finite von Neumann algebra on a Hilbert space H and N a von Neumann subalgebra of M . Then, for each natural number n , the following conditions are equivalent:*

- (i) $M_n(M)^+ = C_n(N)$.
- (ii) $M_n(M)^+ = D_n(N)$.

Proof. We may assume that M has a cyclic and separating trace vector ξ_0 . It is known in the theory of standard forms (see [6, Lemma 1.1]) that $P_n = \overline{\text{co}}\{[x_i^* x_j \xi_0] \in M_n(H) | x_i \in M\}$ is a self-dual cone in a Hilbert space $M_n(H)$; that is, the dual cone $P'_n = \{\xi \in M_n(H) | (\xi, \eta) \geq 0 \text{ for all } \eta \in P_n\}$ of P_n coincides with itself. Put $Q_n = \overline{\text{co}}\{[a_i^* c^* c a_j \xi_0] | a_i \in N, c \in M\}$ and $R_n = \{[\xi_{i,j}] \in M_n(H) | \sum_{i,j=1}^n a_i J a_j J \xi_{i,j} \in P \text{ for every } a_1, \dots, a_n \in N\}$, where J denotes an isomorphic involution and P a self-dual cone in H related to ξ_0 . We then have $Q_n \subset P_n \subset R_n$, $Q_n \subset Q'_n$, and $Q'_n = R_n$. One easily sees that $Q_n = P_n$ implies $R_n = P_n$. Conversely, $R_n = P_n$ implies $Q_n = P_n$ by using the Hahn-Banach Theorem. Hence we have the desired result by virtue of the following facts: For $x_{i,j} \in M$ and $a_i \in M$, we have

$$\sum_{i,j=1}^n a_i J a_j J x_{i,j} \xi_0 = \sum_{i,j=1}^n a_i x_{i,j} a_j^* \xi_0.$$

If $[x_{i,j} \xi_0]$ belongs to P_n for $x_{i,j} \in M$, then for $y_i \in M'$

$$([x_{i,j}][y_i \xi_0], [y_i \xi_0]) = ([x_{i,j} \xi_0], [y_i^* y_j \xi_0]) \geq 0$$

because $\overline{\text{co}}\{[y_i^* y_j \xi_0] | y_i \in M'\}$ is a dual cone of P_n , which means $[x_{i,j}] \geq 0$.

Lemma 5. Let A be an abelian von Neumann algebra.

(i) If M is a von Neumann algebra and N is a von Neumann subalgebra of M such that $M_n(M)^+ = C_n(N)$ for some $n \in \mathbb{N}$, then $M_n(M \otimes A)^+ = C_n(N \otimes I)$.

(ii) If M is a von Neumann algebra on a separable Hilbert space H and N is a von Neumann subalgebra of M such that $M_n(M)^+ = D_n(N)$ for some $n \in \mathbb{N}$, then $M_n(M \otimes A)^+ = D_n(N \otimes I)$.

Proof. (i) We identify $M_n(M \otimes A) = (M \otimes A) \otimes M_n$ with $(M \otimes M_n) \otimes A$. Since A is abelian, the positive cone $((M \otimes M_n) \otimes A)^+$ coincides with the strongly closed convex hull of the elements $x \otimes y$ for $x \in (M \otimes M_n)^+$ and $y \in A^+$.

In fact, suppose that there exists an element z_0 in $((M \otimes M_n) \otimes A)^+$ which does not belong to the cone $\overline{\text{co}}^{\sigma\text{-}w}\{x \otimes y | x \in (M \otimes M_n)^+, y \in A^+\}$. By the Hahn-Banach Theorem, there exists a σ -weakly continuous linear functional ϕ on $(M \otimes M_n) \otimes A$ such that $\phi(z_0) < 0$ and $\phi(z) \geq 0$, where z is any element of the above convex hull. However, ϕ must be a positive functional on $(M \otimes M_n) \otimes A$ by [7, Theorem 3.4], which is a contradiction. By the Kaplansky density theorem, the set of all elements $x \otimes y$ for $x \in (M \otimes M_n)$ and $y \in A^+$ is strongly dense in $((M \otimes M_n) \otimes A)^+$.

Thus we have $((M \otimes M_n) \otimes A)^+ = \overline{\text{co}}^s\{[x_i^* x_j] \otimes a^* a | x_i \in M, a \in A\}$. By assumption, we have

$$\begin{aligned} M_n(M \otimes A)^+ &= \overline{\text{co}}^s\{[y_i^* c^* c y_j] \otimes a^* a | y_i \in N, c \in M, a \in A\} \\ &= \overline{\text{co}}^s\{[y_i \otimes 1]^*(c \otimes a)^*(c \otimes a)(y_j \otimes 1) | y_i \in N, c \in M, a \in A\}. \end{aligned}$$

(ii) Take an element $[x_{i,j}]$ in $D_n(N \otimes I)$. By identifying A with $L^\infty(\Gamma, \mu)$, where Γ is a locally compact Hausdorff space and μ is a positive Radon measure in Γ , each $x_{i,j} \in M \otimes A$ is expressed as

$$x_{i,j} = \int_{\Gamma}^{\oplus} x_{i,j}(\gamma) d\mu(\gamma),$$

where $x_{i,j}(\gamma)$ is an element of M . We then have, for any $a_i \in N$,

$$\int_{\Gamma}^{\oplus} \sum_{i,j=1}^n a_i^* x_{i,j}(\gamma) a_j d\mu(\gamma) = \sum_{i,j=1}^n (a_i^* \otimes 1) x_{i,j} (a_j \otimes 1) \geq 0.$$

Since H is separable, we have $\sum_{i,j=1}^n a_i^* x_{i,j}(\gamma) a_j \geq 0$ locally μ -almost everywhere by [8, Proposition IV.7.12]. Since N is separable in the strong topology, the above inequality holds for all $a_i \in N$. By assumption, we have $x_{i,j}(\gamma) \geq 0$ locally μ -almost everywhere. It follows that

$$\begin{aligned} ([x_{i,j}][\xi_i], [\xi_i]) &= \sum_{i,j=1}^n (x_{i,j} \xi_j, \xi_i) \\ &= \int_{\Gamma}^{\oplus} \sum_{i,j=1}^n (x_{i,j}(\gamma) \xi_j(\gamma), \xi_i(\gamma)) d\mu(\gamma) \geq 0, \end{aligned}$$

for any $[\xi_i] \in (H \otimes L^2(\Gamma, \mu)) \otimes H_n$, of which each component is expressed as

$$\xi_i = \int_{\Gamma}^{\oplus} \xi_i(\gamma) d\mu(\gamma), \quad \xi_i(\gamma) \in H.$$

Therefore, $[x_{i,j}] \geq 0$. This completes the proof.

We shall next consider, in the matrix algebra M , the condition for N that the cone $D_n(N)$ includes properly $M_n(M)^+$.

Lemma 6. *If $k < m$ then $M_n(M_{km})^+ \neq D_n(M_k \otimes I_m)$ for $n \geq 2$.*

Proof. We first consider the case $n = 2$. Suppose that $[x_{i,j}]$ ($x_{i,j} \in M_{km} = M_k \otimes M_m$, $i, j = 1, 2$) belongs to $D_2(M_k \otimes I_m)$ and $k < m$. We then have, for $a_i \in M_k$ and $\xi \in H_{km}$,

$$\begin{aligned} &\left(\begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{bmatrix} \begin{bmatrix} (a_1 \otimes 1_m)\xi \\ (a_2 \otimes 1_m)\xi \end{bmatrix}, \begin{bmatrix} (a_1 \otimes 1_m)\xi \\ (a_2 \otimes 1_m)\xi \end{bmatrix} \right) \\ &= \left(\sum_{i,j=1}^2 (a_i^* \otimes 1_m) x_{i,j} (a_j \otimes 1_m) \xi, \xi \right) \geq 0. \end{aligned}$$

Put $a_i = [\alpha_{p,q}^{(i)}] \in M_k$ for $i = 1, 2$ and

$$\xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_m \end{bmatrix}, \quad \xi_j = \begin{bmatrix} \lambda_1^{(j)} \\ \vdots \\ \lambda_m^{(j)} \end{bmatrix},$$

for $\alpha_{p,q}^{(i)} \in \mathbb{C}$ and $\lambda_r^{(j)} \in \mathbb{C}$. We then obtain the following equality:

$$\begin{aligned} (a_i \otimes 1_m) \xi &= \begin{bmatrix} a_i & & 0 \\ & \ddots & \\ 0 & & a_i \end{bmatrix} \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_m \end{bmatrix} \\ &= \sum_{q=1}^k {}^t[\alpha_{1,q}^{(i)} \lambda_q^{(1)}, \dots, \alpha_{k,q}^{(i)} \lambda_q^{(1)}, \dots, \alpha_{1,q}^{(i)} \lambda_q^{(m)}, \dots, \alpha_{k,q}^{(i)} \lambda_q^{(m)}] \end{aligned}$$

for $i = 1, 2$. Put

$$\eta_q = {}^t[\alpha_{1,q}^{(1)}, \alpha_{2,q}^{(1)}, \dots, \alpha_{k,q}^{(1)}, \alpha_{1,q}^{(2)}, \alpha_{2,q}^{(2)}, \dots, \alpha_{k,q}^{(2)}] \in H_{2k}$$

and

$$\zeta_q = {}^t[\lambda_q^{(1)}, \lambda_q^{(2)}, \dots, \lambda_q^{(m)}] \in H_m,$$

for $q = 1, 2, \dots, k$. If we consider that $[x_{i,j}]$ acts on $H_{2k} \otimes H_m$, we get the inequality $\omega(\eta_1 \otimes \zeta_1 + \eta_2 \otimes \zeta_2 + \dots + \eta_k \otimes \zeta_k, \eta_1 \otimes \zeta_1 + \eta_2 \otimes \zeta_2 + \dots + \eta_k \otimes \zeta_k)([x_{i,j}]) \geq 0$, where $\omega_{\xi, \eta}$ is denoted by $\omega(\xi, \eta)$. Conversely, if an element $[x_{i,j}]$ of $M_2(M_{km})$ satisfies the above inequality for every $\eta_q \in H_{2k}$ and $\zeta_q \in H_m$ ($q = 1, 2, \dots, k$), $[x_{i,j}]$ belongs to $D_2(M_k \otimes I_m)$.

Now, suppose that any element of $D_2(M_k \otimes I_m)$ is positive. By identifying M_{2km} with $M_{2k} \otimes M_m$, the order of $M_{2k} \otimes M_m$ is determined by the family of positive elements of order k in $(M_{2k} \otimes M_m)^*$.

In fact, we have

$$\begin{aligned} & \omega(\eta_1 \otimes \zeta_1 + \dots + \eta_k \otimes \zeta_k, \eta_1 \otimes \zeta_1 + \dots + \eta_k \otimes \zeta_k) \\ &= \sum_{s,t=1}^k \omega(\eta_s, \eta_t) \otimes \omega(\zeta_s, \zeta_t). \end{aligned}$$

If c_i is an arbitrary element of M_{2k} for $i = 1, 2, \dots, k$, then we have

$$\sum_{s,t=1}^k \omega(\eta_t, \eta_s)(c_s^* c_t) = \left(\sum_{s=1}^k c_s \eta_s, \sum_{s=1}^k c_s \eta_s \right) \geq 0.$$

Hence $[\omega(\eta_t, \eta_s)]$ is a positive element of $M_k(M_{2k})^*$. Similarly, $[\omega(\zeta_t, \zeta_s)]$ is a positive element of $M_k(M_m)^*$.

By [9, Theorem 2.2], M_{2k} or M_m is k -homogeneous, which is a contradiction. We obtained the required results.

We next consider the case $n \geq 3$. By the above argument there exists an element $y = [y_{i,j}] \in D_2(M_k \otimes I_m)$ which does not belong to $M_2(M_{km})^+$. Put $Y = [Y_{i,j}] = \begin{bmatrix} y & 0 \\ 0 & E \end{bmatrix}$, where $Y_{i,j} \in M_{km}$ and $E = I_{km} \oplus \dots \oplus I_{km} \in M_{n-2}(M_{km})$. If $a_i \in M_k \otimes I_m$ for $i = 1, 2, \dots, n$, then we have

$$\sum_{i,j=1}^n a_i^* Y_{i,j} a_j = \sum_{i,j=1}^2 a_i^* y_{i,j} a_j + \left(\sum_{i=3}^n a_i \right)^* \left(\sum_{i=3}^n a_i \right) \geq 0.$$

But Y is not a positive matrix. This completes the proof.

We now obtain the following two theorems by using the above discussion, in which we investigate the factorization of positive cones of order n of a finite von Neumann algebra of type I and an injective factor.

Theorem 7. Let M be a finite von Neumann algebra of type I expressed as $M = \sum_{i=1}^{m \oplus} M_{k(i)} \otimes A_{k(i)}$ with $A_{k(i)}$ abelian von Neumann algebras. Put $N = \sum_{i=1}^{m \oplus} N_{k(i)} \otimes I_{k(i)}$ with $N_{k(i)}$ von Neumann subalgebras of $M_{k(i)}$. Then, for each natural number $n \geq 2$, the following conditions are equivalent:

- (i) $M_n(M)^+ = C_n(N)$.
- (ii) $M_n(M)^+ = D_n(N)$.
- (iii) Every $N_{k(i)}$ has a cyclic vector.

Proof. (i) $\Rightarrow$ (ii) We may only consider the case $M = M_k \otimes A$. Since A is abelian, $M_n(M)^+$ coincides with the strongly closed convex hull of all elements $[x_{i,j}] \otimes a$ with $[x_{i,j}] \in M_n(M_k^+)$ and $a \in A^+$ by the proof of Lemma 5. On the other hand, we have

$$\begin{aligned} C_n(N_k \otimes I) &= \overline{\text{co}}^s \{ [(x_i \otimes 1)^* c^* c (x_j \otimes 1)] | x_i \in N_k, c \in M_k \otimes A \} \\ &= \overline{\text{co}}^s \{ [x_i^* y^* y x_j] \otimes a | x_i \in N_k, y \in M_k, a \in A^+ \}. \end{aligned}$$

By assumption, we get the equality $M_n(M_k)^+ = C_n(N_k)$. By Lemmas 4 and 5, we have $M_n(M_k \otimes A_k)^+ = D_n(N_k \otimes I)$.

(ii) $\Rightarrow$ (iii) Let N be a von Neumann subalgebra of a matrix algebra M_k and N have no cyclic vector. It then suffices to prove $M_n(M_k)^+ \neq D_n(N)$ for $n \geq 2$. We may consider N as $M_m \otimes I_{m'} \oplus B$ ($\subset M_k$) with $m < m'$, where B is a von Neumann algebra on $H_{k-mm'}$. By Lemma 6 we can find an element $[x_{i,j}]$ in $M_n(M_m \otimes M_{m'})$ such that $[x_{i,j}] \in D_n(M_m \otimes I_{m'})$ and $[x_{i,j}]$ is not positive. Put $X_{i,j} = x_{i,j} \oplus \delta_{ij} 1_B \in M_k$. If $a_i \in M_{mm'}$ and $b_i \in B$ for $i = 1, 2, \dots, n$, then

$$\sum_{i,j=1}^n (a_i \oplus b_i)^* X_{i,j} (a_j \oplus b_j) = \sum_{i,j=1}^n a_i^* x_{i,j} a_j \oplus \sum_{i=1}^n b_i^* b_i.$$

Then $[X_{i,j}]$ belongs to $D_n(N)$, but it is not positive.

(iii) $\Rightarrow$ (i) We apply Proposition 2 and Lemmas 4 and 5.

Theorem 8. *Let H be a separable Hilbert space.*

(i) *If M is an injective von Neumann algebra on H , then there exists an abelian $*$ -subalgebra A of M such that $M_n(M)^+ = C_n(A)$ for every $n \in \mathbb{N}$.*

(ii) *If M is an injective factor (or semifinite, injective von Neumann algebra), then there exists an abelian $*$ -subalgebra A of M such that $M_n(M)^+ = D_n(A)$ for every $n \in \mathbb{N}$.*

Proof. The statement (i) is due to [4, Proposition 2.2].

(ii) Suppose that M is an injective factor on a separable Hilbert space H . M is then a Krieger's factor (cf. [2, Theorems 7.5, 7.7; 3, Corollary 2.4]). By [1, Theorem 1], there exists an increasing sequence $\{\varepsilon_k\}$ of normal conditional expectations of M onto finite-dimensional subalgebras N_k with $M = (\bigcup_{k=1}^{\infty} N_k)^{-s}$. Then, for every $x \in M$, $\varepsilon_k(x)$ converges strongly to x by [1, Lemma 2]. Fix a natural number k . Since N_k is decomposed into a finite direct sum of matrix algebras, we construct a maximal abelian $*$ -subalgebra A_k of N_k , each component of which consists of all diagonal matrices. We then have $(M_n(N_k))^+ = D_n(A_k)$ by Proposition 2. Since all maximal abelian $*$ -subalgebras of a finite-dimensional von Neumann algebra are spatially isomorphic, there exists a maximal abelian $*$ -subalgebra A_{k+1} of N_{k+1} such that $A_k \subset A_{k+1}$ and $M_n(N_{k+1})^+ = D_n(A_{k+1})$. By induction, we put $A = \bigcup_{k=1}^{\infty} A_k$, which is an abelian $*$ -subalgebra of M . Take an element $[x_{i,j}]$ in $D_n(A)$. We then have, for each k ,

$$\sum_{i,j=1}^n a_i^* \varepsilon_k(x_{i,j}) a_j = \varepsilon_k \left(\sum_{i,j=1}^n a_i^* x_{i,j} a_j \right) \geq 0,$$

for all $a_i \in A_k$. Hence $[\varepsilon_k(x_{i,j})] \geq 0$. As k tends to ∞ , we obtain $[x_{i,j}] \geq 0$.

Suppose now that M is a semifinite, injective von Neumann algebra. We may only consider that M is continuous. By using the reduction theory, M is written as

$$M = R_0 \otimes B_0 \oplus R_0 \otimes B_1 \otimes B(K),$$

where R_0 is a hyperfinite factor of type II_1 and B_0 and B_1 are abelian von Neumann algebras because of the unicity of R_0 . By the first half of the proof, we can find abelian von Neumann subalgebras E and F of R_0 and $B(K)$ with $M_n(R_0)^+ = D_n(E)$ and $M_n(B(K))^+ = D_n(F)$, respectively. Put $A = E \otimes I \oplus E \otimes I \otimes F$. We then have, by Lemma 5, $M_n(M)^+ = D_n(A)$. This completes the proof.

We note here that a maximal abelian von Neumann subalgebra on H having a cyclic vector is maximal abelian in $B(H)$ (see, for example, [5, Proposition 2.8.3]). The above theorem shows that there exists a $*$ -subalgebra N of an injective factor M of type II or III satisfying the conditions (i) and (ii) and having no cyclic vector.

Finally, we shall state the following remark for a completely positive map. Recall that a positive linear map ϕ of a von Neumann algebra M_1 to M_2 is said to be completely positive if, for every natural number n , a multiplicity map $\phi \otimes 1_n$ maps a positive cone in $M_n(M_1)$ into a positive cone in $M_n(M_2)$.

Remark 9. Let M_1 and M_2 be von Neumann algebras and N_1 and N_2 subalgebras of M_1 and M_2 satisfying, for all natural numbers n , $C_n(N_1) = M_n(M_1)^+$ and $D_n(N_2) = M_n(M_2)^+$, respectively. Suppose that ϕ is a normal positive map of M_1 into M_2 . Then ϕ is completely positive if and only if ϕ satisfies, for all natural numbers n , the relation

$$\sum_{i,j=1}^n b_i^* \phi(a_i^* c^* c a_j) b_j \geq 0,$$

for every $a_1, \dots, a_n \in N_1$, $c \in M_1$, and $b_1, \dots, b_n \in N_2$.

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