COMPLETELY POSITIVE PROJECTIONS
ON A HILBERT SPACE

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Abstract. The purpose of this paper is to prove that a completely positive projection on a Hilbert space associated with a standard form of a von Neumann algebra induces the existence of a conditional expectation of the von Neumann algebra with respect to a normal state, and we consider the application to a standard form of an injective von Neumann algebra.

Introduction

Many authors have studied the problem of how an algebraic structure of a von Neumann algebra is determined by the underlying Hilbert space. Connes [C1] introduced an orientation of a homogeneous selfdual cone in a Hilbert space and characterized the Hilbert space associated with a standard form of a von Neumann algebra. A geometric interpretation was given by Iochum [I1] to an algebraic notion of a conditional expectation (i.e., a projection of norm one) of a von Neumann algebra by using an orientation property in a selfdual cone.

On the other hand, Schmitt and Wittstock [SW] characterized a matrix-ordered Hilbert space with a family of selfdual cones arising from standard forms of von Neumann algebras by a projection property instead of orientation. Matrix-ordered spaces were first introduced by Choi and Effros [CE] as the appropriate objects to which completely positive maps apply and enabled us to handle non-commutative order. In [M] the author proved that a faithful normal conditional expectation with respect to a cyclic and separating vector on a von Neumann algebra induces the existence of a completely positive projection on the Hilbert space, and considered an approximation property in $L^2(M)$ with respect to a completely positive map when $M$ is an injective factor. In the present note we shall consider the converse theorem.

We shall use the book of Takesaki [T2] as a reference of the standard results of operator algebras. We shall use the notion as introduced in [SW] with respect to matrix-ordered standard forms and their construction.

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Results

Throughout this section we assume that \((M, H, H_n^+, n \in \mathbb{N})\) is a matrix-ordered standard form [SW, Chapter 1] of a \(\sigma\)-finite \(W^*\)-algebra \(M\). Let \(e\) be a completely positive projection on \(H\), i.e., \(e_n = e \otimes 1_n\) maps \(H_n^+\) into \(H_n^+\) for every \(n \in \mathbb{N}\) where \(1_n\) denotes the identity on the \(n \times n\) matrices. We set \(K = eH, K_n = e_nH_n, K_n^+ = e_nH_n^+, n \in \mathbb{N}\).

We need two lemmata to prove the main theorem.

**Lemma 1.** If \((K, K_n^+, n \in \mathbb{N})\) is as above, then there exists a von Neumann algebra \(N\) such that \((N, K, K_n^+, n \in \mathbb{N})\) is a matrix-ordered standard form.

**Proof.** It is straightforward to see that \((K, K_n^+, n \in \mathbb{N})\) is a matrix-ordered Hilbert space with selfdual cones. It suffices by [SW, Theorem 4.3] to prove that any closed face of each selfdual cone \(K_n^+ (n \in \mathbb{N})\) is projectable. Let \(F\) be a closed face of \(K_n^+\). The face generated by \(F\) in \(H_n^+\) is denoted by \(\langle F \rangle\). We then obtain by [I2, II.1.7 Lemma and II.1.3 Proposition] and [C1, Theorem 4.2] the inclusion

\[
P_FK_n^+ = e_nP_{\langle F \rangle}e_nK_n^+ \subset e_nP_{\langle F \rangle}H_n^+ = e_nP_{F^+}H_n^+ = e_nF^{-1} = e_n\overline{\langle F \rangle} = F,
\]

where \(P_F\) (resp. \(P_{\langle F \rangle}\)) denotes the orthogonal projection of \(K_n\) (resp. \(H_n\)) onto the closed subspace generated by \(F\) (resp. \(\langle F \rangle\)). \(\square\)

**Lemma 2.** Let \(M, e\) and \(N\) be as above. Assume, in addition, that \(e\xi_0 = \xi_0\) for some cyclic and separating vector \(\xi_0\) in \(H^+\) for \(M\). If we put \(L = M \cap \{e\}'\), then \(L|eH = eM|eH = N\).

**Proof.** It is easy to see that \(Je = eJ, eJ|K = J_{K^+}, J_ne_n = e_nJ_n\) and \(e_nJ_ne_n|K_n = J_{K_n^+}\).

We shall first prove that \(eM|K \subset N\). Take a derivation \(\delta\) in \(D(H_2^+)\). We see from [C1, Lemma 5.3] that \(e_2\delta e_2J_2 = J_2e_2\delta e_2\) and if \(\eta, \zeta \in K_2^+\) and \((\eta, \zeta) = 0\), then \((e_2\delta e_2\eta, \zeta) = (\delta\eta, \zeta) = 0\). This implies that \(e_2\delta(K_2)\) belongs to \(D(K_2^+)\). By virtue of the standard form \((M_2(M), H_2, J_2, H_2^+)\) we see from [C1, Theorem 3.4 c)] that for each element \(X = [a\ b] \in M_2(M)\) there exists \(Y = [c\ d] \in M_2(N)\) satisfying \(e_2(X + J_2X J_2)\Xi = (Y + J_{K_2^+}Y J_{K_2^+})\Xi\) for all \(\Xi \in K_2\). By setting \(\Xi = [0\ c\ 0\ 0]\) with \(\xi \in K\) we have

\[
\begin{bmatrix}
0 & e\xi \\
0 & 0
\end{bmatrix} = \begin{bmatrix}
JbJ\xi & (a + Jd)\xi \\
0 & c\xi
\end{bmatrix},
\]

so that \(b = c = 0\) and \(e\xi = a\xi + Jd\xi\). Moreover, if we set \(\Xi = [0\ 0]\), then \([0\ 0] = [0\ (d + Jd)\xi]\). It follows that \(e\xi = a\xi - d\xi\) for all \(\xi \in K\), whence \(eM|K \subset N\).

We next want to prove that \(N \subset L|K\). Take any projection \(p\) in \(N\). Since in general by [C1, Theorem 4.2] the map \(q \rightarrow qJqH^+\) is an order isomorphism of the set of all projections of \(M\) onto the set of all closed faces of \(H^+\), we see that \([0\ 1]J_{K_2^+}[0\ 1]J_{K_2^+}\) is a closed face of \(K_2^+\), which will be denoted by \(F\), and \(P_F = [0\ 1]J_{K_2^+}[0\ 1]J_{K_2^+}\). Therefore, there exists a projection \(P = [a\ b] \in M_2(M)\) such that \(P_{\langle F \rangle} = P_{\langle F \rangle}\Xi\) for all \(\Xi \in K_2\). By setting \(\Xi = [0\ 0]\) we have

\((*)\quad p\xi = e\xi JcJ\xi\)
for all $\xi \in K$. On the other hand, since $e_2 P_F e_2 \leq P(F)$ we have for all $\xi \in K$
\[
\begin{bmatrix}
0 & 0 \\
0 & \xi
\end{bmatrix} = \begin{bmatrix}
p & 0 \\
0 & 1
\end{bmatrix} J_{K_2^+} \begin{bmatrix}
p & 0 \\
0 & 1
\end{bmatrix} J_{K_2^+} \begin{bmatrix}
0 & 0 \\
0 & \xi
\end{bmatrix} = \begin{bmatrix}
a & b \\
b^* & c
\end{bmatrix} J_2 \begin{bmatrix}
a & b \\
b^* & c
\end{bmatrix} J_2 \begin{bmatrix}
0 & 0 \\
0 & \xi
\end{bmatrix} = \begin{bmatrix}
b J b J \xi & b J c J \xi \\
c J b J \xi & c J c J \xi
\end{bmatrix}.
\]
We then have by [SW, Corollary 3.3] $b \xi = 0$. Since $K^+$ contains by assumption a cyclic and separating vector for $M$, we have $b = 0$. It follows that both $a$ and $c$ must be projections because $P$ is a projection. Since
\[\xi = c J c J \xi = e \xi\]
for all $\xi \in K$, we have by assumption $c = 1$. Hence the equality (*) implies $p \xi = e a \xi$ for all $\xi \in K$. We see from [I2, II.1.7 Lemma] that $e_2 P(F) = P(F) e_2$, i.e. $[e_2 J_0 J e_2] = [e_2 J_0 J e_2]$, whence $e a = e c$. Therefore, for every element $x$ in $N$ there exists an element $y$ in $M$ commuting with $e$ such that $x \xi = y \xi$ for all $\xi \in K$.

Consequently, the argument above shows that

\[L|K \subset e M|K \subset N \subset L|K.\]

**Theorem 3.** Let $(M, H, H^+_n, n \in \mathbb{N})$ be a matrix-ordered standard form of the von Neumann algebra $M$, and let $e$ be a completely positive projection on $H$ satisfying $e \xi_0 = \xi_0$ for some cyclic and separating vector $\xi_0 \in H^+$ for $M$. If $L = M \cap \{e\}'$, then $(L|e H, e H, e H^+_n, n \in \mathbb{N})$ is a matrix-ordered standard form. In addition, there exists a faithful normal conditional expectation $\xi$ of $M$ onto $L$ such that $\omega_{\xi_0} \xi = \omega_{\xi_0}$. Furthermore, we have $L|e H = e M|e H$.

**Proof.** This part of the proof is due to Iochum [I1, Theorem 3.1.6]. For any element $x$ in $M$ there exists by Lemma 2 $\alpha(x)$ in $L$ such that $e x \xi = \alpha(x) \xi$ for all $\xi \in K$. $\alpha(x)$ is uniquely determined since $e \xi_0 = \xi_0$ is a separating vector for $M$. We then have for all $y$ in $L$

\[(e x^* \xi_0, J y J \xi_0) = (\xi_0, x J y J \xi_0) = (\xi_0, e x J y J \xi_0) = (\alpha(x)^* \xi_0, J y J \xi_0).
\]

Hence $e x^* \xi_0 = \alpha(x)^* \xi_0$ because of the density of $J L J \xi_0$ in $K$. It follows that

\[e S x \xi_0 = e x^* \xi_0 = e \alpha(x)^* \xi_0 = S \alpha(x) \xi_0 = S e \xi_0.
\]

Since $M \xi_0$ is a core set of $S$, we have $e S = S e$. Hence $\Delta_{\xi_0} e = e \Delta_{\xi_0} = \xi_0$ and $L \xi_0$ is invariant under $\Delta_{\xi_0}^t$ ($\forall t \in \mathbb{R}$). From the theorem of Takesaki [T1, Theorem] we see the existence of the conditional expectation $\xi$. This completes the proof.

Note that in general $e$ does not belong to $M$. As an application of the above theorem we immediately obtain the following corollary in which we consider a von Neumann algebra from the point of view of semidiscreteness in the related $L^2$-space.

**Corollary 4.** Let $(M, H, H^+_n, n \in \mathbb{N})$ be a matrix-ordered standard form of the von Neumann algebra $M$. If there exists an increasing sequence $\{e^{(n)}\}$ of completely positive projections of finite rank on $H$ which converges strongly to $1$ such that $e^{(1)} \xi_0 = \xi_0$ for some cyclic and separating vector $\xi_0 \in H^+$ for $M$, then $M$ is injective.
Proof. By assumption we have for all \( n \in \mathbb{N} \) \( e^{(n)}\xi_0 = e^{(1)}e^{(1)}\xi_0 = e^{(1)}\xi_0 = \xi_0 \). It follows from Theorem 3 that for each \( n \) there exists a finite-dimensional von Neumann subalgebra \( L_n \) of \( M \) satisfying \( L_n|e^{(n)}H = e^{(n)}M|e^{(n)}H \). Since \( L_n\xi_0 = e^{(n)}M\xi_0 \) and \( \xi_0 \) is a separating vector for \( M \), we have \( L_n \subseteq L_{n+1} \). Thus we have \( M = \{ \bigcup_n L_n \}^{-s} \). So \( M \) is injective.

We have many results for injectivity in the theory of operator algebras (cf. for example, [CE], [C2]). In [S] Schmitt studied the Arveson space, which is a Banach space having the completely positive extension property with predual being a certain matrix-ordered Banach space, via several equivalent properties which are the finite injectivity, the approximative factorization property, the matricial Riesz interpolation property and the matricial Hahn-Banach property, and gave the characterization of matrix-ordered standard forms of injective \( W^* \)-algebras.

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