Material Design for Reduction of Thermal Stress in a Functionally Graded Material Rotating Disk*

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The purpose of this paper is to present a method of material design for the weight reduction, the high thermal radiation and the relaxation of in-plane thermal stress and centrifugal stress in a rotating disk without hole composed of functionally graded material (FGM) with arbitrary thermal and mechanical nonhomogeneities in the radial direction. The disk is subjected to an intermittent heating in an annular region near the outer radius and has an arbitrary variation of heat-transfer coefficient along the radial position on the upper and lower surfaces. The transient temperature field is analyzed by modifying Vodiccka's method for one-dimensional boundary value problems in composite regions, and the thermal stress and centrifugal stress are respectively obtained by solving approximately the equilibrium equations expressed in terms of the displacement component as Euler's differential equations. The material design is carried out for the rotating disks composed of Ti+6Al+4V and SUS410 FGM, and Ti+6Al+4V, PSZ and SUS410 FGM.

Key Words: Functionally Graded Materials, Material Design, Stress Relaxation, Thermal Stress, Nonhomogeneous Rotating Disk, Centrifugal Stress, Intermittent Heating, Analytical Solution

1. Introduction

The material and strength design problems of rotating disks, accompanied by analysis of temperature fields and thermal stress fields, are of great importance for disk brake systems⁽¹⁾, circular saw tensioning for woodworking⁽²⁾, grinding wheels⁽³⁾, end face cutting by using lathes, etc., and many studies on this subject have been conducted. It is experimentally indicated that when repeated overloads are applied to a rotor of one-piece disk brake system for large motorcycles, bowl-shaped deformation due to the yield phenomenon occurs at the bridges between the holes

for thermal radiation and weight reduction⁽¹⁾. In order to avoid this yield, the design without holes using the material with light weight and superior heat radiation characteristics is required. However such a design with any homogeneous materials is impossible. Therefore, it is tried to use functionally graded materials (FGMs) with nonhomogeneity in the radial direction which have light weight, superior strength and heat radiation characteristics, for the large disk brake system without holes. In fact, it begins to be studied using FGMs for the brake rotors of aircrafts and a new type of bullet train subjected to more severer thermal cycle loads.

The goal of our study is to design a disk brake rotor without holes by using FGMs, which has high thermal radiation, strength and light weight characteristics of the same level as the existing disk brakes. As a first step, this paper proposes an analytical technique of the temperature field in a FGM disk with arbitrary thermal and mechanical nonhomogeneities in the radial direction subjected to a thermal cycle load which simulates that of a disk brake, and a method of material design for relaxation of the associated thermal and centrifugal stresses. In this analysis, variations in heat-transfer coefficient along the radial

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direction on the upper and lower surfaces are taken into account, because it has been indicated in our previous reports^{(4),(5)} that the spatial change in heat-transfer coefficient sufficiently influences the transient temperature and thermal stress distributions.

2. Analytical Solution for Transient Temperature Field

2.1 Temperature field due to a uniform heat load

Consider the transient temperature field in a thin non-homogeneous disk composed of FGM with thickness b , inner radius r_0 and outer radius r_n . The FGM disk has the heat-transfer coefficients expressed in arbitrary functions, $h = h_u(r)$ and $h = h_l(r)$ varying with the radial coordinate on the upper and lower surfaces, and uniform heat generation q per unit time and unit volume in the annular region of the vicinity of the outer radius ($r_N \leq r \leq r_n$). The governing equation of heat conduction in such annular disk having the nonhomogeneities only in the radial direction is expressed as follows:

$$\frac{1}{r} (r\lambda(r)T_{,r})_{,r} - \frac{h_u(r) + h_l(r)}{b} (T - T_\infty) + qH(r - r_N) = \rho(r)c(r)T_{,t}, \quad (1)$$

where $\lambda(r)$, $\rho(r)$ and $c(r)$ are thermal conductivity, density and specific heat, respectively, and $,r$ means a partial differentiation with respect to r . $H(\cdot)$ denotes the Hevi-side's step function. It assumes that the initial temperature and the surrounding medium temperature on the upper and lower surfaces are T_∞ , and the heat-transfer coefficients and the surrounding medium temperatures on the both lateral surfaces of $r = r_0$ and $r = r_n$ are given by h_{in} , $\theta_{in}(t)$ and h_{out} , $\theta_{out}(t)$, respectively. It is impossible to obtain an analytical solution to Eq. (1) for arbitrary nonhomogeneities and arbitrary variations of heat-transfer coefficients in the radial direction. Therefore, the disk is first divided into n homogeneous layers which have distinct, constant thermal properties as shown in Fig. 1, and then the temperature field in the multilayer disk is analyzed based on the

Fig. 1 Analytical model for a FGM rotating disk

modified Vodiccka's method.

The thermal conductivity, specific heat, density, outer radius, and temperature function of the i th layer are denoted by λ_i , c_i , ρ_i , r_i , and T_i , respectively. Then the transient heat conduction problem is formulated as follows:

$$\lambda_i \left(T_{i,rr} + \frac{1}{r} T_{i,r} \right) - \frac{h_{ui} + h_{li}}{b} T_i + Q(r) = \rho_i c_i T_{i,t} \quad (i = 1, 2, \dots, n), \quad (2)$$

$$\lambda_1 T_{1,r} - h_{in}(T_1 - \theta_{in}^*(t)) = 0 \quad (r = r_0), \quad (3)$$

$$\lambda_n T_{n,r} + h_{out}(T_n - \theta_{out}^*(t)) = 0 \quad (r = r_n), \quad (4)$$

$$T_i^* = T_{i+1}^*, \quad \lambda_i T_{i,r} = \lambda_{i+1} T_{i+1,r} \quad (r = r_i), \quad (5)$$

$$T_i^* = 0 \quad (i = 1, 2, \dots, n) \quad (t = 0), \quad (6)$$

where

$$T_i^* = T_i - T_\infty, \quad \theta_{in}^*(t) = \theta_{in}(t) - T_\infty,$$

$$\theta_{out}^*(t) = \theta_{out}(t) - T_\infty, \text{ and } Q(r) = qH(r - r_N).$$

As well as the transient temperature solution to the functionally graded plate⁽⁶⁾, the solution to one-dimensional transient heat-conduction problem above could be separated as follows:

$$T_i^*(r, t) = \sum_{m=1}^{\infty} \varphi_m(t) X_{im}(r) + \sum_{j=1}^2 L_{ij}(r) P_j(t), \quad (7)$$

where

$$P_1(t) = -\theta_{in}^*(t), \quad P_2(t) = \theta_{out}^*(t), \quad (8)$$

and $L_{ij}(r)$ is the solution to the following differential equation:

$$L_{ij,rr} + \frac{1}{r} L_{ij,r} - \frac{(B_{ui} + B_{li})}{r_n^2 b \lambda_i} L_{ij} = 0. \quad (9)$$

The constants, B_{ui} and B_{li} in Eq. (9) are Biot numbers on the upper and lower surfaces given by $B_{ui} = h_{ui} r_n / \lambda_0$ and $B_{li} = h_{li} r_n / \lambda_0$, respectively, with a representative thermal conductivity λ_0 . Additionally, $\bar{b} = b / r_n$ and $\bar{\lambda}_i = \lambda_i / \lambda_0$. Furthermore, $X_{im}(r)$ is the solution to the eigenvalue problem as follows:

$$\begin{cases} X_{im}(r)_{,rr} + \frac{1}{r} X_{im}(r)_{,r} + \left(\frac{\gamma_m^2}{\kappa_i} - \frac{(B_{ui} + B_{li})}{r_n^2 b \lambda_i} \right) X_{im}(r) = 0 \\ X_{1m,r} - H_0 X_{1m} = 0 & (r = r_0) \\ X_{nm,r} + H_n X_{nm} = 0 & (r = r_n) \\ X_{im} = X_{(i+1)m}, \quad \lambda_i X_{im,r} = \lambda_{i+1} X_{(i+1)m,r} & (r = r_i) \end{cases} \quad (10)$$

where $\kappa_i = \lambda_i / c_i \rho_i$, $H_0 = h_{in} / \lambda_1$, and $H_n = h_{out} / \lambda_n$.

$L_{ij}(r)$ and $X_{im}(r)$ are given by the following equations, respectively:

$$L_{ij}(r) = C_{ij} I_0(g_i r) + D_{ij} K_0(g_i r), \quad (11)$$

$$X_{im}(r) = A_{im} J_0(d_{im} r) + B_{im} Y_0(d_{im} r), \quad (12)$$

where $I_0(\cdot)$ and $K_0(\cdot)$ are the zero-order modified Bessel functions of the first and second kind, respectively, $J_0(\cdot)$ and $Y_0(\cdot)$ are the zero-order Bessel functions of the first

and second kind, respectively, and g_i and d_{im} are given by the following equations:

$$g_i = \sqrt{(B_{ui} + B_{li})/r_n^2 \bar{b} \lambda_i},$$

$$d_{im} = \sqrt{\gamma_m^2 / \kappa_i - (B_{ui} + B_{li})/r_n^2 \bar{b} \lambda_i} \quad (i = 1, 2, \dots, n)$$

The eigenvalue function of Eq. (12) is used for the case of $\gamma_m^2 / \kappa_i \geq (B_{ui} + B_{li})/r_n^2 \bar{b} \lambda_i$, and for the case of $\gamma_m^2 / \kappa_i \leq (B_{ui} + B_{li})/r_n^2 \bar{b} \lambda_i$, the following equation is used:

$$X_{im}(r) = A_{im} I_0(d_{im}^* r) + B_{im} K_0(d_{im}^* r), \quad (13)$$

where

$$d_{im}^* = \sqrt{(B_{ui} + B_{li})/r_n^2 \bar{b} \lambda_i - \gamma_m^2 / \kappa_i}.$$

The unknown coefficients A_{im} and B_{im} are normalized as follows:

$$A_{im} = \bar{A}_{im} A_{nm}, \quad B_{im} = \bar{B}_{im} A_{nm}, \quad \bar{A}_{nm} = 1,$$

where $\bar{A}_{im}$ and $\bar{B}_{im}$ are determined by Eq. (10).

Substituting Eq. (7) into Eqs. (3) to (5) and using Eqs. (8) and (10), the following conditions concerning with $L_{ij}(r)$ are obtained:

$$\begin{cases} L_{11,r} - H_0 L_{11} = H_0 & (r = r_0) \\ L_{n1,r} + H_n L_{n1} = 0 & (r = r_n) \\ L_{i1} = L_{(i+1)1}, \quad \lambda_i L_{i1,r} = \lambda_{i+1} L_{(i+1)1,r} & (r = r_i) \end{cases} \quad (14)$$

$$\begin{cases} L_{12,r} - H_0 L_{12} = 0 & (r = r_0) \\ L_{n2,r} + H_n L_{n2} = H_n & (r = r_n) \\ L_{i2} = L_{(i+1)2}, \quad \lambda_i L_{i2,r} = \lambda_{i+1} L_{(i+1)2,r} & (r = r_i) \end{cases} \quad (15)$$

The eigenvalues γ_m are determined based on a condition that the unknown coefficients A_{im} and B_{im} have nonzero solutions, and are given as positive roots of the following transcendental equation:

$$G_{e1} \cdot E_{e1} \cdot E_{e2} \cdots E_{en-2} \cdot E_{en-1} \cdot a_{en} = 0, \quad (16)$$

where

$$\begin{cases} a_{en} = \begin{bmatrix} A_{nm} \\ B_{nm} \end{bmatrix}, \quad G_{e1} = [g_{11} \quad g_{21}] \\ g_{11} = -d_{1m} J_1(d_{1m} r_0) - H_0 J_0(d_{1m} r_0) \\ g_{21} = -d_{1m} Y_1(d_{1m} r_0) - H_0 Y_0(d_{1m} r_0) \\ E_{ei} = C_{ei}^{-1} \cdot D_{ei+1} \\ C_{ei} = \begin{bmatrix} J_0(d_{im} r_i) & Y_0(d_{im} r_i) \\ -\lambda_i d_{im} J_1(d_{im} r_i) & -\lambda_i d_{im} Y_1(d_{im} r_i) \end{bmatrix} \\ D_{ei+1} = \begin{bmatrix} J_0(d_{i+1m} r_i) & Y_0(d_{i+1m} r_i) \\ -\lambda_{i+1} d_{i+1m} J_1(d_{i+1m} r_i) & -\lambda_{i+1} d_{i+1m} Y_1(d_{i+1m} r_i) \end{bmatrix} \end{cases} \quad (17)$$

The summation of Eq. (7) is taken over infinite positive roots of Eq. (16).

The eigenfunction $X_{im}(r)$ which is the solution of Eq. (10), satisfies the following orthogonal relationship for discontinuous weight functions:

$$\sum_{i=1}^n \int_{r_{i-1}}^{r_i} \frac{\lambda_i}{\kappa_i} r X_{im}(r) X_{ik}(r) dr = \begin{cases} \text{const.} & (m=k) \\ 0 & (m \neq k) \end{cases} \quad (18)$$

From Eqs. (6) and (7), the following equation is defined:

$$G_i(r) = \sum_{m=1}^{\infty} \varphi_m(0) X_{im}(r) = - \sum_{j=1}^2 L_{ij}(r) P_j(0). \quad (19)$$

With the orthogonal relationship of Eq. (18), $G_i(r)$, $L_{ij}(r)$, and $Q(r)$ can be expressed in the following series forms:

$$G_i(r) = \sum_{m=1}^{\infty} g_m X_{im}(r), \quad L_{ij}(r) = \sum_{m=1}^{\infty} l_{mj} X_{im}(r),$$

$$\frac{\kappa_i}{\lambda_i} Q(r) = \sum_{m=1}^{\infty} q_m X_{im}(r), \quad (20)$$

where

$$\begin{cases} g_m = M_m^{-1} \sum_{i=1}^n \frac{\lambda_i}{\kappa_i} \int_{r_{i-1}}^{r_i} r G_i(r) X_{im}(r) dr \\ l_{mj} = M_m^{-1} \sum_{i=1}^n \frac{\lambda_i}{\kappa_i} \int_{r_{i-1}}^{r_i} r L_{ij}(r) X_{im}(r) dr \\ q_m = M_m^{-1} \sum_{i=1}^n \int_{r_{i-1}}^{r_i} r Q(r) X_{im}(r) dr \\ M_m = \sum_{i=1}^n \frac{\lambda_i}{\kappa_i} \int_{r_{i-1}}^{r_i} r [X_{im}(r)]^2 dr \end{cases} \quad (21)$$

Substituting Eqs. (7) and (20) into Eq. (2) and using Eqs. (9) and (10), the following differential equation to $\phi_m(t)$ can be obtained:

$$\varphi_m(t)_{,t} + \gamma_m^2 \varphi_m(t) = q_m - \sum_{j=1}^2 l_{mj} P_j(t)_{,t}. \quad (22)$$

Furthermore, from Eqs. (19) and (20), the initial condition of function $\phi_m(t)$ is:

$$\varphi_m(0) = g_m. \quad (23)$$

The solution $\phi_m(t)$ from Eqs. (22) and (23) can be obtained as follows:

$$\varphi_m(t) = e^{-\gamma_m^2 t} \left\{ g_m + \int_0^t e^{\gamma_m^2 t'} \left(q_m - \sum_{j=1}^2 l_{mj} \frac{dP_j(t')}{dt'} \right) dt' \right\}. \quad (24)$$

By substituting Eqs. (11) and (12) into Eq. (7), the desired temperature function $T_i^*(r, t)$ can be obtained as follows:

$$T_i^*(r, t) = \sum_{m=1}^{\infty} \varphi_m(t) A_{nm} \{ \bar{A}_{im} J_0(d_{im} r) + \bar{B}_{im} Y_0(d_{im} r) \}$$

$$+ \sum_{j=1}^2 \{ C_{ij} I_0(g_i r) + D_{ij} K_0(g_i r) \} P_j(t), \quad (25)$$

where for the case of $\gamma_m^2 / \kappa_i \leq (B_{ui} + B_{li})/r_n^2 \bar{b} \lambda_i$, Eq. (13) is used instead of Eq. (12), and $P_j(t)$ and $\phi_m(t)$ are given by Eqs. (8) and (24), respectively.

2.2 Temperature field due to an intermittent heat load

The transient temperature function $\hat{T}_i(r, t)$ in the FGM disk subjected to an intermittent heat load (Fig. 2) which simulates a thermal cycle load, is derived by using the temperature function, Eq. (25) in the i th layer of the FGM

Fig. 2 Intermittent heat load

disk with the constant heat load. The temperature function $\hat{T}_i(r, t)$ in the i th layer during the first cooling period t_1' to t_2 can be obtained as follows:

$$\hat{T}_i(r, t) = T_i^{*1}(r, t) - T_i^{*1}(r, t - t_1'), \quad (26)$$

where $T_i^{*1}(r, t)$ is the temperature function in the FGM disk with the initial temperature $T_i^{*1}(r, t_1')$ at $t = t_1'$ and a uniform heat generation q in the annular region ($r_N \leq r \leq r_n$), and $-T_i^{*1}(r, t - t_1')$ is the temperature function in the FGM disk with initially zero temperature at $t = t_1'$ and a uniform heat sink $-q$ in the same region. The superscript 1 of $T_i^{*1}(r, t)$ indicates the cycle number of the intermittent heat load and $T_i^{*1}(r, t_1')$ is obtained by substituting $t = t_1'$ into $T_i^*(r, t)$ of Eq. (25). Furthermore, the temperature function during the second heating period t_2 to t_2' , can be expressed by the following equation:

$$\hat{T}_i(r, t) = T_i^{*1}(r, t) - T_i^{*1}(r, t - t_1') + T_i^{*2}(r, t - t_2). \quad (27)$$

Similarly, the temperature function during the k th heating period t_k to t_k' is obtained as follows:

$$\hat{T}_i(r, t) = \hat{T}_i^*(r, t) + \hat{T}_i^{*k}(r, t - t_k), \quad (28)$$

and the temperature function during the k th cooling period t_k' to t_{k+1} is expressed as follows:

$$\hat{T}_i(r, t) = \hat{T}_i^*(r, t) + T_i^{*k}(r, t - t_k) - T_i^{*k}(r, t - t_k'), \quad (29)$$

where $\hat{T}_i^*(r, t)$ denotes the temperature of the FGM disk before the k th heating period.

3. Stress Analysis in a Rotating FGM Disk Subjected to the Transient Temperature Field

The analytical solution is approximately derived for the plane stress problem in a thin nonhomogeneous annular disk, which is subjected to the axisymmetrical transient temperature field mentioned above and rotates with a constant angular velocity ω . The desired stress components are obtained by a superposition of the two stress fields as follows:

$$\sigma_{rr} = \sigma_{rr}^1 + \sigma_{rr}^2, \quad \sigma_{\theta\theta} = \sigma_{\theta\theta}^1 + \sigma_{\theta\theta}^2, \quad (30)$$

where σ_{ij}^1 is the thermal stress and σ_{ij}^2 is the centrifugal stress of the rotating disk.

Firstly, the Hooke's law for a plane thermal stress problem is expressed in terms of the displacement in the radial direction U_r^1 as follows:

$$\begin{cases} \sigma_{rr}^1 = \frac{E}{1-\nu^2} (U_{r,r}^1 + \nu r^{-1} U_r^1) - \frac{E}{1-\nu} \alpha T^* \\ \sigma_{\theta\theta}^1 = \frac{E}{1-\nu^2} (r^{-1} U_r^1 + \nu U_{r,r}^1) - \frac{E}{1-\nu} \alpha T^* \end{cases} \quad (31)$$

where $E = E(r)$ and $\alpha = \alpha(r)$ are Young's modulus and the coefficient of linear thermal expansion, respectively which are both arbitrary functions of the radial coordinate, and Poisson's ratio ν is assumed to be constant. T^* denotes the temperature change expressed by $T - T_\infty$. Because the influence of the temperature dependence of the Young's modulus and the coefficient of linear thermal expansion on the elastic thermal stress has been reported to be small by the present authors⁽⁵⁾, the temperature dependence of their material properties can be neglected.

Next, consider a plane stress problem in a FGM disk with uniform thickness which is rotating with a constant angular velocity around the axis. The Hooke's law for this problem is expressed in terms of the displacement in the radial direction U_r^2 as follows:

$$\begin{cases} \sigma_{rr}^2 = \frac{E}{1-\nu^2} (U_{r,r}^2 + \nu r^{-1} U_r^2) \\ \sigma_{\theta\theta}^2 = \frac{E}{1-\nu^2} (r^{-1} U_r^2 + \nu U_{r,r}^2) \end{cases} \quad (32)$$

By substituting Eqs. (31) and (32) into the equilibrium equations, the differential equations for the radial displacement U_r^1 and U_r^2 are obtained as follows:

$$\begin{aligned} & U_{r,rr} + f_1(r)r^{-1}U_{r,r} + g_1(r)r^{-2}U_r \\ &= \begin{cases} (1+\nu)\left\{\frac{E_{,r}}{E}\alpha T^* + \alpha_{,r}T^* + \alpha T_{,r}^*\right\}, & (U_r = U_r^1) \\ -\frac{w\omega^2 r}{Eg}(1-\nu^2), & (U_r = U_r^2) \end{cases}, \end{aligned} \quad (33)$$

where $w = w(r)$, ω , and g are the specific weight, angular velocity, and gravitational acceleration, respectively, and

$$f_1(r) = rE_{,r}/E + 1, \quad g_1(r) = \nu rE_{,r}/E - 1. \quad (34)$$

It is impossible to derive an exact solution for a linear second-order differential equation with arbitrary variable coefficients as Eq. (33), which is obtained for Young's modulus of FGM. Therefore, the disk is divided into n layers in the same way as the temperature analysis, and $f_1(r)$ and $g_1(r)$ in each layer are approximated by distinct constants f_{1j} and g_{1j} ($j = 1, 2, \dots, n$). Thus, the displacements in the j th layer U_{rj}^1 and U_{rj}^2 satisfy the following Euler's differential equation:

$$\begin{aligned} & U_{rj,rr} + f_{1j}r^{-1}U_{rj,r} + g_{1j}r^{-2}U_{rj} \\ &= \begin{cases} (1+\nu)\left\{\frac{E_{,r}}{E}\alpha T^* + \alpha_{,r}T^* + \alpha T_{,r}^*\right\}, & (U_{rj} = U_{rj}^1) \\ -\frac{w\omega^2 r}{Eg}(1-\nu^2), & (U_{rj} = U_{rj}^2) \end{cases}. \end{aligned} \quad (35)$$

The general solutions for these equations are obtained as follows:

$$U_{rj}^1 = C_{1j}r^{\xi_{j1}} + C_{2j}r^{\xi_{j2}}$$

$$+ \sum_{k=1}^2 (-1)^{k-1} \frac{1+\nu}{\xi_{j1}-\xi_{j2}} r^{\xi_{jk}} \int_{r_0}^r r^{1-\xi_{jk}} \left\{ \frac{E_r}{E} \alpha T^* + \alpha_{,r} T^* + \alpha T^*_{,r} \right\} dr, \quad (36)$$

$$U_{rj}^2 = C_{1j}^* r^{\xi_{j1}} + C_{2j}^* r^{\xi_{j2}} + \frac{\omega^2(1-\nu^2)}{g(\xi_{j2}-\xi_{j1})} \sum_{k=1}^2 r^{\xi_{jk}} (-1)^k \int_{r_0}^r \frac{w r^{2-\xi_{jk}}}{E} dr, \quad (37)$$

where ξ_{j1} and ξ_{j2} are the distinct real roots of the following equation:

$$x^2 + (f_{1j}-1)x - g_{1j} = 0. \quad (38)$$

Finally, the desired expressions for the stress components are obtained by substituting Eqs. (36) and (37) into Eqs. (31) and (32) as follows:

$$\sigma_{\theta\theta j}^1 = \frac{E}{1-\nu^2} \sum_{k=1}^2 (1+\xi_{jk}\nu) r^{\xi_{jk}-1} \times \left[C_{kj} + (-1)^{k-1} \frac{1+\nu}{\xi_{j1}-\xi_{j2}} \int_{r_0}^r r^{1-\xi_{jk}} \times \left\{ \frac{E_r}{E} \alpha T^* + \alpha_{,r} T^* + \alpha T^*_{,r} \right\} dr \right] - \frac{E}{1-\nu^2} \alpha T^*, \quad (39)$$

$$\sigma_{\theta\theta j}^2 = \frac{E}{1-\nu^2} \sum_{k=1}^2 (1+\xi_{jk}\nu) r^{\xi_{jk}-1} \times \left\{ C_{kj}^* + (-1)^{k-1} \frac{\omega^2(1-\nu^2)}{(\xi_{j2}-\xi_{j1})g} \int_{r_0}^r \frac{w r^{2-\xi_{jk}}}{E} dr \right\}, \quad (40)$$

and the expressions for σ_{rrj}^1 and σ_{rrj}^2 can be obtained by replacing $(1+\xi_{jk}\nu)$ in the above expressions for $\sigma_{\theta\theta j}^1$ and $\sigma_{\theta\theta j}^2$ with $(\xi_{jk}+\nu)$. For the case of the rotating annular disk with the fixed inner surface and the stress-free outer surface, the boundary condition and the continuity condition for the stress field are expressed as follows:

$$\begin{cases} U_{r1} = 0 & (r=r_0) \\ \sigma_{rrn} = 0 & (r=r_n) \\ \sigma_{rrj} = \sigma_{rrj+1}, \quad U_{rj} = U_{rj+1} & (r=r_j) \end{cases} \quad (41)$$

The unknown coefficients C_{1j} , C_{2j} , C_{1j}^* , and C_{2j}^* in Eqs. (39) and (40) are determined from Eq. (41).

4. Numerical Results and Discussion

Assuming that each heating interval and cooling interval is given by constants t_h and t_c , respectively, the following dimensionless quantities are introduced:

$$\begin{cases} \zeta = r/r_n, \quad \tau = \kappa_0 t/r_n^2, \quad \tau_n = \kappa_0 t_h/r_n^2 \\ \tau_c = \kappa_0 t_c/r_n^2, \quad \bar{T}_i(\zeta, \tau) = T_i(r, t)/T_{\infty} \\ B_{in} = h_{in} r_n/\lambda_0, \quad B_{out} = h_{out} r_n/\lambda_0 \\ \bar{\sigma}_{ij}^1 = \sigma_{ij}^1/E_0 \alpha_0 T_{\infty} \\ \bar{\sigma}_{ij}^2 = \sigma_{ij}^2/g/w_0 \omega^2 r_n^2 \\ \bar{Q} = Q r_n^2/T_{\infty} \lambda_0, \end{cases} \quad (i, j = r, \theta) \quad (42)$$

where B_{in} and B_{out} are Biot numbers, κ_0 , E_0 , α_0 , and w_0 are the representative values of the thermal diffusivity,

Young's modulus, coefficient of linear thermal expansion and specific weight, respectively, and T_{∞} is the steady-state temperature of the surrounding medium on the outer surface. Furthermore, the surrounding medium temperatures on the inner and outer surfaces are expressed as follows:

$$\begin{cases} \theta_{in}^*(t) = \theta_{in}(t) - T_{\infty} = T_{1\infty}(1 - e^{-t/\tau_1}) - T_{\infty} \\ \theta_{out}^*(t) = \theta_{out}(t) - T_{\infty} = T_{n\infty}(1 - e^{-t/\tau_n}) - T_{\infty} \end{cases} \quad (43)$$

Additionally, $\sum_{m=1}^{\infty}$ in Eq. (25) is truncated as $\sum_{m=1}^{30}$.

The radial variation in the heat-transfer coefficient is assumed to be same on the upper and lower surfaces, and is derived in the same way as our previous report⁽⁵⁾ from the distribution of the local heat transfer coefficient of the air on a disk rotating with 180 rpm, reported by Hirose, et al⁽⁷⁾. Numerical calculations are carried out for the following parameters which simulate the disk shape and the heat load condition of Ref. (1),

$$\begin{cases} \bar{b} = 0.05, \quad T_{\infty}/T_{n\infty} = 0, \quad T_{1\infty}/T_{n\infty} = 0, \quad B_{in} = B_{out} = 1 \\ \tau_x = \kappa_0 t_x/r_n^2 = 1, \quad \tau_s = \kappa_0 t_s/r_n^2 = 1, \quad \bar{Q} = 200 \\ r_N/r_n = 0.73, \quad \tau_h = \tau_c = 1, \quad n = 20, \quad \nu = 0.3 \\ r_0/r_n = 0.1, \quad E_0 \alpha_0 T_{n\infty}/(w_0 \omega^2 r_n^2/g) = 60 \end{cases} \quad (44)$$

The sliding surface (from now on called "flange part") of the disk, contacting with the brake pad becomes high temperature. First, for inner annular region than this part, titanium alloy Ti+6Al+4V which has smaller density than SUS410⁽¹⁾ used for conventional brake disks, is employed for weight reduction. Thus the disk is designed as a Ti+6Al+4V/SUS410 FGM disk. However, no sufficient effect of thermal stress relaxation was obtained. So a design using SUS410/PSZ FGM for the flange part and Ti+6Al+4V for the inner annular region was tried. The reasons of employing PSZ are the following: it has high temperature resistance and it is the toughest ceramic; its coefficient of linear thermal expansion is smaller than SUS410 and therefore the mismatch of coefficients of linear thermal expansion can become small at the interface of titanium alloy and SUS410 bonded; and the weight reduction is simultaneously accomplished. The material properties used here are shown in Table 1.

4.1 A design using Ti+6Al+4V and SUS410

The volume fraction distribution of SUS410 is shown in Fig. 3. As shown in this figure, the numerical calculation was conducted for the four types of graded material composition: linear, Ti-rich, Ti-poor, and bimaterial. The material properties of FGM layer are determined based on the linear mixture rule.

Figures 4–6 show the temperature distribution and the radial and circumferential thermal stress distributions at $\tau = 9$, respectively. Its time corresponds to the time that the 5th heating just finishes. These results show that the effect of thermal stress relaxation by using

Table 1 Material properties⁽⁸⁾⁻⁽¹⁰⁾

	SUS410	PSZ	Ti+6Al+4V
Thermal conductivity λ [W/(m · K)]	24.91	1.67	8.0
Young's modulus E [GPa]	200	211	109
Coefficient of linear thermal expansion $\alpha \times 10^{-6}$ [1/K]	11.7	2.93	8.40
Density ρ [kg/m ³]	7750	5730	4430
Specific heat c [kJ/(kg · K)]	0.461	0.467	0.510

Fig. 3 Volume fraction distribution for a design with two materials

Ti+6Al+4V/SUS410 FGM is small compared with that by using the simple bimaterial of Ti+6Al+4V and SUS410.

Table 2 shows relative weights compared with the weight of homogeneous SUS410 disk. The FGM disk is not light compared with the disk having eyebrow-shaped holes.

4.2 A design using Ti+6Al+4V and PSZ/SUS410 FGM

The material design as shown in Fig. 7 was tried in order to reduce the weight and to relax the large tensile stress on the inner radius and the large compressive stress at the flange part. Titanium alloy was employed for the inner annular region of the flange part, and the graded layer composed of SUS410 and PSZ was used for the flange part. The two volume fraction distributions of PSZ as Fig. 7 were considered. These correspond to the ceramic-rich composition which contains more PSZ than SUS410 (dashed and dotted line) and the ceramic-poor composition which contains less PSZ (dashed line), respectively. In order to reduce the difference of the coefficients of linear thermal expansion at the interface between titanium alloy and PSZ/SUS410 FGM, the volume fraction at the interface was taken to be 60% of SUS410 and 40% of PSZ.

Fig. 4 Transient temperature distribution for Ti+6Al+4V/SUS410 FGM

Fig. 5 Transient thermal stress distribution for Ti+6Al+4V/SUS410 FGM

Fig. 6 Transient thermal stress distribution for Ti+6Al+4V/SUS410 FGM

Table 2 Comparison of weight

Composition pattern	Relative weight
SUS410 homo.	1.0
SUS410 homo. (※)	0.85
Bimaterial	0.85
Linear	0.94
Ti-6Al-4V rich	0.89
Ti-6Al-4V poor	0.98

(※) A disk having eyebrow-shaped holes

Fig. 7 Volume fraction distribution for a design with three materials

Figure 8 shows the temperature distribution at $\tau = 9$. This figure shows that by the addition of PSZ having the smaller thermal conductivity than SUS410, the temperature in the inner annular region where titanium alloy is used decreases, while the maximum temperature at the flange part increases. From the viewpoint of the strength in the inner annular region, the thermal resistance was improved.

Figures 9 and 10 show the distributions of the radial and circumferential thermal stresses at $\tau = 9$, respectively. As the volume fraction of ceramic increases, the relaxation of the tensile, radial thermal stress increases. The tensile thermal stress on the inner radius for the case of ceramic rich was reduced by 42% compared with that for the bimaterial. The compressive circumferential thermal stress at the flange part which causes the yield phenomenon, was maximum for the bimaterial. The maximum compressive thermal stress for the case of the ceramic rich was reduced by 38% compared with that for the bimaterial.

Table 3 shows a comparison of relative weight for each composition pattern. It is shown in Tables 2 and 3 that the addition of ceramic yields the lighter weight than that for the bimaterial, and that the weight of the ceramic rich disk was reduced by 12% compared with that for the conventional disk with eyebrow-shaped holes.

Fig. 8 Transient temperature distribution for Ti+6Al+4V and PSZ/SUS410 FGM

Fig. 9 Transient thermal stress distribution for Ti+6Al+4V and PSZ/SUS410 FGM

Fig. 10 Transient thermal stress distribution for Ti+6Al+4V and PSZ/SUS410 FGM

Figures 11 and 12 show the distributions of the radial and circumferential stress caused by the centrifugal force, respectively. No significant difference in the maximum

Table 3 Comparison of weight

Composition pattern	Relative weight
Ceramic rich	0.75
Ceramic poor	0.81

Fig. 11 Centrifugal stress distribution for Ti+6Al+4V and PSZ/SUS410 FGM

Fig. 12 Centrifugal stress distribution for Ti+6Al+4V and PSZ/SUS410 FGM

value of tensile radial stress due to the difference in the composition pattern of PSZ/SUS410 FGM is found. This is because the material composition of the inner annular region is the same. It is noted from Fig. 12 that there is a large discontinuity of tensile circumferential stress due to the discontinuous difference in density at the interface between Ti+6Al+4V and SUS410/PSZ FGM layers, and that the maximum value of tensile circumferential stress for the ceramic rich is 8% smaller than that for the homogeneous SUS410.

5. Conclusions

For the material design applying FGM to disk brake rotors of large motorcycles, analytical methods for a tran-

sient heat conduction problem and a plane thermal stress problem in an annular disk made of FGM due to an intermittent heat load were proposed. In these analyses, the FGM disk with arbitrary nonhomogeneities in the radial direction was approximated as a multilayer disk composed of many distinct homogeneous layers, and the spatial variation in the heat-transfer coefficient on the upper and lower surfaces was taken into account. Furthermore, numerical calculations were carried out for two kinds of rotating FGM disks, and the influence of difference in composition pattern on the reduction of weight and the relaxation of thermal stress and centrifugal stress was discussed. Thus the following results were obtained:

- (1) a design using Ti+6Al+4V and SUS410, the effect of thermal stress relaxation in SUS410/Ti+6Al+4V FGM disks is small compared with that of the bimaterial disk composed of SUS410 and Ti+6Al+4V;
- (2) a design using Ti+6Al+4V and SUS410/PSZ FGM,
 - 1) the temperature in the region of the titanium alloy becomes lower compared with that for the bimaterial disk, while the temperature in the flange part becomes higher;
 - 2) for the case of the flange part with the ceramic rich composition pattern, the maximum tensile thermal stress at the inner radius, the maximum compressive thermal stress at the outer radius and the maximum tensile stress due to the centrifugal force are sufficiently relaxed, and the weight reduction is also realized.

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