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Mathematical treatment for the oscillation of the solution of the equation for Foucault's pendulum

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1 Introduction

We shall consider the 4-dimensional non-linear system

$$\dot{x} = Ax + f(x) \tag{1}$$

where $x = x(t)$ is the 4-dimensional vector-valued function of t, $x = (x_k(t))$, $1 \leq k \leq 4$, $\dot{x} = \left(\dfrac{d}{dt}x_k(t)\right)$, $1 \leq k \leq 4$, $|x|$ the norm of x, for example $|x| = \displaystyle\sum_{k=1}^{4} |x_k|$, $f(x)$ is differential with respect to x,

$$|f(x)| = o(|x|) \qquad as\ |x| \to 0 \tag{2}$$

and moreover A is the constant 4×4 matrix, $A = (a_{ij})_{1\leq j\leq 4}^{1\leq i\leq 4}$, and $|A|$ the operator norm with respect to $|x|$. As is stated in the below, the equation for Foucault's pendulum may be written in the form of (1) and (2), and furthermore the following assertion is made in the arguments of its physics.

If every eigenvalue of A is not a real number and if $|x(0)|$ is sufficiently small and not zero, then each component $x_k(t)$ of solution $x(t)$ oscillates. Although this assertion is seemed in the physics to be treated with no problem, we propose to investigate it from the view of mathematics.

Now, the equation for Foucault's pendulum is the following

$$\begin{cases} \ddot{x} + ax = b\dot{y} - \frac{x}{l^2}v \\ \ddot{y} + ay = -b\dot{x} - c\dot{z} - \frac{y}{l^2}v \end{cases} \tag{3}$$

where t is time, x and y are functions of t, $\cdot$ denotes the derivative with respect to t, v is defined by (4)

$$\begin{cases} v = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 + b(x\dot{y} - \dot{x}y) + c(\dot{y}z - y\dot{z}) - cl\dot{y} - gz \\ z = l - \sqrt{l^2 - x^2 - y^2}\ , \qquad\qquad \dot{z} = \frac{x\dot{x}+y\dot{y}}{l-z} \end{cases} \tag{4}$$

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and g is the gravitational constant, l the length of the string of the pendulum, $a = \frac{g}{l}$, b and c are positive constants determined from the latitude on the earth to the position of the supporting point of the pendulum. Our initial condition is that $x(0) \neq 0$ and $x(0) = y(0) = \dot{y}(0) = 0$, and it is a feature that $y(t)$ begins to oscillate as t goes on, which exhibits the rotation of the earth.

When we set that $x = (x, \dot{x}, y, \dot{y})$, the linear part of (3) may be written in the following system

$$\dot{x} = Ax \tag{5}$$

where

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -a & 0 & 0 & b \\ 0 & 0 & 0 & 1 \\ 0 & -b & -a & 0 \end{pmatrix} \tag{6}$$

and $f(x)$ comes from v.
The assertion made in the arguments of physics is the following : (3) is reduced to (5) by the neglect of higher order terms of $\frac{x}{l}$ and $\frac{y}{l}$ when $\frac{x}{l}$ and $\frac{y}{l}$ are sufficiently small, and since every eigenvalue of A of (6) is distinct and pure imaginary number, $x(t)$ and $y(t)$ of non-identically zero solutions of (5) oscillate. Therefore $x(t)$ and $y(t)$ of non-identically zero solutions of (3) with small initial value oscillate (see [3], [4], [5], [6]).

We think that the oscillations of solutions of (5) do not necessarily imply the oscillations of solutions of (3) with small initial value, and as our conclusion of the investigation that we additionally require the stability for identically zero solution of (3), which may guarantee for the above reduction to hold for all $t > 0$ and result the infinitely many times oscillations of $x(t)$ and $y(t)$, respectively.

We shall state our results of Theorems 1, 2 and 3 in below, whose proofs will be shown in the section 2, respectively. Through this paper we may assume that A satisfy the following conditions (i) and (ii) :

(i) every eigenvalue of A is not a real number

(ii) A cannot be taken to the following form by any permutation of $1 \leq i$, $j \leq 4$

$$A = \begin{pmatrix} A_1 & 0 \\ A_2 & A_3 \end{pmatrix}$$

where A_1, A_2, A_3 and O are 2×2 matrices and O is the zero matrix.

Theorem 1
Let $u(t) = e^{At}u(0)$, where $u(0)$ is the 4-dimensional nonzero vector. Then each component $u_k(t)$ of $u(t)$, $1 \leq k \leq 4$, is not identically zero and oscillates infinitely many times in the manner such that $u_k(t)$ changes its sign on the

interval $[nT, (n+1)T]$, where T is a positive constant and n is any positive number, that is, there exist at least two numbers t_1 and t_2, $nT \le t_1 \le t_2 \le (n+1)T$, such that

$$u_k(t_1)u_k(t_2) < 0 \tag{7}$$

Remark 1
In the proof of Theorem 1, two kind of T's are given depending on the multiplicity of eigenvalues of A.

Theorem 2
Let $x(t)$ be a solution of (1) with initial condition $x(0)$ which is not zero and sufficiently small. Then each component $x_k(t)$ of $x(t)$ for $1 \le k \le 4$, changes its sign on the interval $0 \le t \le T$, where T is the constant of Theorem 1, that is , there exist at least two numbers t_1 and t_2, $0 \le t_1 \le t_2 \le T$, such that

$$x_k(t_1)x_k(t_2) < 0 \tag{8}$$

Moreover, if the identically zero solution of (1) is stable, then each component $x_k(t)$ of $x(t)$ oscillates infinitely many times in the manner as in (7).

Remark 2
The stability of identically zero solution of (1) is not seemed to be known because the eigenvalues of A of (6) are purely imaginary numbers.

Next we shall treat the damped case of (3) such that

$$\begin{aligned} \ddot{x} + k\dot{x} + ax &= b\dot{y} - \tfrac{x}{l^2}v \\ \ddot{y} + k\dot{y} + ay &= -b\dot{x} - c\dot{z} - \tfrac{y}{l^2}v \end{aligned} \tag{9}$$

where k is a positive constant.

Theorem 3
Assume that (10) and (11) hold

$$a \; < \; 4 \tag{10}$$

$$k^2 \; > \; \frac{\sqrt{K^2 + a(4-a)(2a+b)^2} - K}{4-a} \tag{11}$$

where $K = 2a + 2b^2 - ab^2 - 2a^2$. Then the identically zero solution of (9) is asymptotically stable. Consequently if $|x(0)| + |\dot{x}(0)| + |y(0)| + |\dot{y}(0)|$ is not zero and sufficiently small, then the both of $x(t)$ and $y(t)$ oscillate infinitely many times in manner of (7), respectively.

Remark 3
Since $a = \frac{g}{l}$, (10) means that l is sufficiently large compared with g.

2 Proofs of Theorem 1, 2 and 3

2.1 Proof of Theorem 1

Clearly $u(t)$ satisfies

$$\dot{u}(t) = Au(t)$$

which may be written in the following such that for $1 \le k \le 4$,

$$\dot{u}_k(t) = \sum_{j=1}^{4} a_{kj} u_j(t) \tag{12}$$

Now we shall prove that

$$u_k(t) \not\equiv 0 \qquad \text{for } 1 \le k \le 4 \tag{13}$$

On the contrary suppose that some componentof $u(t)$ is identically zero, say

$$u_1(t) \equiv 0 \tag{14}$$

If $a_{12} = a_{13} = a_{14} = 0$, then a_{11} must be the real eigenvalue of A, and hence a_{1k} is not zero for some $2 \le k \le 4$, which may be written by a permutation of $\{1, 2, 3, 4\}$ that

$$a_{12} \ne 0 \tag{15}$$

and moreover by a suitable choice of the unit of t that

$$a_{12} = 1 \tag{16}$$

It follow from (12) that

$$u_2 = -a_{13}u_3 - a_{14}u_4 \tag{17}$$

and moreover that u_3 and u_4 satisfy the 2-dimensional system

$$\begin{pmatrix} \dot{u}_3 \\ \dot{u}_4 \end{pmatrix} = B \begin{pmatrix} u_3 \\ u_4 \end{pmatrix} \tag{18}$$

where B is the 2×2 matrix, $B = (b_{ij})_{1 \le j \le 2}^{1 \le i \le 2}$, and

$$\begin{aligned} b_{11} &= a_{33} - a_{32}a_{13} \qquad & b_{12} &= a_{34} - a_{32}a_{14} \\ b_{21} &= a_{43} - a_{42}a_{13} \qquad & b_{22} &= a_{44} - a_{42}a_{14} \end{aligned}$$

Now we shall show that

$$u_3 \equiv u_4 \equiv 0 \tag{19}$$

If (19) does not hold, then the two eigenvalues of B are not real numbers. In fact, in the case of (i) where eigenvalues of A are simple complex numbers, we may write these eigenvalues in form such that $\lambda_1 = \alpha_1 + i\beta_1$ and $\lambda_2 = \alpha_2 + i\beta_2$, where $\alpha_1, \alpha_2, \beta_1$ and β_2 are real numbers, $\beta_1 \neq 0$, $\beta_2 \neq 0$ and either $\alpha_1 \neq \alpha_2$ or $\beta_1 \neq \beta_2$. Therefore the argument of the Jordan canonical form of A shows that $u_3(t)$ and $u_4(t)$ take the form such that

$$u_k(t) = e^{\alpha_1 t}(C_{k1}\cos\beta_1 t + D_{k1}\sin\beta_1 t) + e^{\alpha_2 t}(C_{k2}\cos\beta_2 t + D_{k2}\sin\beta_2 t) \quad (20)$$

where C_{k1}, C_{k2}, D_{k1} and D_{k2} are constants for $k = 3$ and for $k = 4$. On the other hand, in the case of (ii) where eigenvalues of A are double and complex numbers, we may write these eigenvalues in the form such that $\lambda = \alpha + i\beta$, where α and β are real numbers and $\beta \neq 0$, and $u_3(t)$ and $u_4(t)$ take the form such that

$$u_k(t) = e^{\alpha t}\{(C_{k1} + tC_{k2})\cos\beta t + (D_{k1} + tD_{k2})\sin\beta t\} \quad (21)$$

where C_{k1}, C_{k2}, D_{k1} and D_{k2} are constants for $k = 3$ and for $k = 4$. In order that (18) has the solution of the form of either (20) or (21), eigenvalues of B must be complex conjugate numbers, say $\lambda = \alpha + i\beta$, where α and β are real numbers and $\beta \neq 0$. Therefore both (20) and (21) are reduced to the form such that

$$u_k(t) = e^{\alpha t}(C_k\cos\beta t + D_k\sin\beta t) \quad (22)$$

where C_k and D_k are constants for $k = 3$ and for $k = 4$. We shall show that $u_3(t) \equiv 0$. From (18) it follows that

$$\dot{u}_3(t) = b_{11}u_3(t) + b_{12}u_4(t) \quad (23)$$

and hence substituting (22) into the right hand side of (23) we obtain that

$$\dot{u}_3(t) = e^{\alpha t}(E_1\cos\beta t + E_2\sin\beta t) \quad (24)$$

where $E_1 = b_{11}C_3 + b_{12}C_4$ and $E_2 = b_{11}D_3 + b_{12}D_4$. On the other hand, it follows from the direct differentiation of $u_3(t)$ of (22) that

$$\dot{u}_3(t) = e^{\alpha t}\{(\alpha C_3 + \beta D_3)\cos\beta t + (\alpha D_3 - \beta C_3)\sin\beta t\} \quad (25)$$

Equating the both of the right hand sides of (24) and (25) we may obtain the equation such that

$$\begin{aligned} \alpha C_3 + \beta D_3 &= E_1 \\ \alpha D_3 - \beta C_3 &= E_2 \end{aligned} \quad (26)$$

Since (26) must hold if β is replaced by $-\beta$, it follows that

$$\begin{aligned} \alpha C_3 - \beta D_3 &= E_1 \\ \alpha D_3 + \beta C_3 &= E_2 \end{aligned} \quad (27)$$

(26) and (27) implies that

$$C_3 = D_3 = 0 \quad (28)$$

which implies that $u_3 \equiv 0$. Similarly we may show that $u_4 \equiv 0$. Above all we obtain that $u_3 \equiv u_4 \equiv 0$, which contradicts to our assumption such that (19) does not hold.

Next we shall show that (7) holds. If (7) does not hold, then we may assume that

$$u_k(t) \geq 0 \quad \text{for} \quad nT \leq t \leq (n+1)T \tag{29}$$

for some $k, 1 \leq k \leq 4$, and for some positive number n. Firstly we shall teat the case where eigenvalues of A are simple, and hence $u_k(t)$ takes the form of (20). In this case we set T to be any number such that

$$T > \frac{\pi}{\beta_1} + \frac{\pi}{\beta_2} \tag{30}$$

where both of β_1 and β_2 in (20) are assumed to be positive. Now we consider the function $w(t)$ such that

$$w(t) = e^{-\alpha_1 t} u_k(t) \tag{31}$$

that is

$$w(t) = C_{k1} \cos \beta_1 t + D_{k1} \sin \beta_1 t + e^{(\alpha_2 - \alpha_1)t}(C_{k2} \cos \beta_2 t + D_{k2} \sin \beta_2 t)$$

(29) and (31) imply that

$$w(t) \geq 0 \quad \text{for} \quad nT \leq t \leq (n+1)T \tag{32}$$

Moreover we see that

$$w(t) + w\Big(t + \frac{\pi}{\beta_1}\Big) = e^{(\alpha_2 - \alpha_1)t} z(t) \tag{33}$$

that is

$$z(t) = C_{k2} \cos \beta_2 t + D_{k2} \sin \beta_2 t + e^{(\alpha_2 - \alpha_1)\frac{\pi}{\beta_1}} \{C_{k2} \cos(\beta_2 t + \frac{\beta_2}{\beta_1}\pi) + D_{k2} \sin(\beta_2 t + \frac{\beta_2}{\beta_1}\pi)\}$$

where (32)implies that

$$z(t) \geq 0 \quad \text{for} \quad nT \leq t \leq nT + T - \frac{\pi}{\beta_1} \tag{34}$$

Since

$$z(t) + z\Big(t + \frac{\pi}{\beta_2}\Big) \equiv 0 \tag{35}$$

it follows from (34) that

$$z(t) \equiv 0 \quad \text{for} \quad nT \leq t \leq nT + T - \frac{\pi}{\beta_1} - \frac{\pi}{\beta_2} \tag{36}$$

Therefore (33) implies that

$$w(t) + w\Big(t + \frac{\pi}{\beta_1}\Big) \equiv 0 \quad \text{for} \quad nT \le t \le nT + T - \frac{\pi}{\beta_1} - \frac{\pi}{\beta_2} \tag{37}$$

and hence (32) that

$$w(t) \equiv 0 \quad \text{for} \quad nT \le t \le nT + T - \frac{\pi}{\beta_1} - \frac{\pi}{\beta_2} \tag{38}$$

which implies together with (31) that

$$u_k(t) \equiv 0 \quad \text{for} \quad nT \le t \le nT + T - \frac{\pi}{\beta_1} - \frac{\pi}{\beta_2} \tag{39}$$

Since $u_k(t)$ is analytic for t and $T > \frac{\pi}{\beta_1} + \frac{\pi}{\beta_2}$, (39) implies that

$$u_k(t) \equiv 0 \tag{40}$$

which contradicts to our first assertion of Theorem 1.

Secondly we shall treat the case of (i) where eigenvalue of A are double complex numbers, and hence $u_k(t)$ takes the form of (21). In this case, we set T to be any number such that

$$T > \frac{\pi}{\beta} \tag{41}$$

where β is the number of (21) and positive. Now we consider

$$w(t) = e^{-\alpha t} u_k(t) \tag{42}$$

this is

$$w(t) = (C_{k1} + tC_{k2}) \cos \beta t + (D_{k1} + tD_{k2}) \sin \beta t \tag{43}$$

(29) and (42) imply that

$$w(t) \ge 0 \quad \text{for} \quad nT \le t \le (n+1)T \tag{44}$$

Since

$$w(t) + w\Big(t + \frac{\pi}{\beta}\Big) \equiv 0 \tag{45}$$

it follows from (44) that

$$w(t) \equiv 0 \quad \text{for} \quad nT \le t \le nT + T - \frac{\pi}{\beta} \tag{46}$$

and hence from (42) that

$$u_k(t) \equiv 0 \quad \text{for} \quad nT \le t \le nT + T - \frac{\pi}{\beta} \tag{47}$$

Since $u_k(t)$ is analytic for t and $T > \frac{\pi}{\beta}$, we may see that

$$u_k(t) \equiv 0$$

which is a contradiction. Thus the proof of Theorem 1 is completed.

2.2 Proof of Theorem 2

Firstly we shall prove the first part of our assertion such that each component $x_k(t)$ changes its sign on the interval $[0, T]$ when $|x(0)|$ is not zero and sufficiently small.

On the contrary suppose that this does not holds, and hence that

$$x_k(t) \geq 0 \qquad \text{for} \quad 0 \leq t \leq T \tag{48}$$

Setting m to be the number such that

$$m = \max_{0 \leq t \leq T} |x(t)|$$

which is positive, we may see from the continuity of solutions of (1) with respect to its initial value $x(0)$ that

$$m \to 0 \qquad \text{as} \quad |x(0)| \to 0$$

and from (2) that

$$\frac{|f(x(t))|}{m} \to 0 \tag{49}$$

uniformly for $0 \leq t \leq T$ as $|x(0)| \to 0$. Setting

$$u(t) = \frac{x(t)}{m}$$

we may obtain the following

$$\dot{u}(t) = Au(t) + \frac{f(x(t))}{m} \tag{50}$$

$$|u(t)| \leq 1 \qquad \text{for} \quad 0 \leq t \leq T \tag{51}$$

$$|u(s)| = 1 \qquad \text{for some}\, s,\ 0 \leq s \leq T \tag{52}$$

We may obtain from (50) that

$$|\dot{u}(t)| < |A| + 1 \qquad \text{for} \quad 0 \leq t \leq T \tag{53}$$

when $|x(0)|$ is sufficiently small. Since (51) and (53) imply that the family $\{u(t)\}$, where m is sufficiently small, is uniformly bounded and equicontinuous on $[0, T]$, and hence we may assume by Ascoli-Arzera's theorem that $\{u(t)\}$ converges uniformly on $[0, T]$ as $|x(0)| \to 0$. Therefore we may set

$$w(t) = \lim_{x(0) \to 0} u(t) \tag{54}$$

and hence it follows from (49) and (50) that

$$\dot{w}(t) = Aw(t) \tag{55}$$

where (52) implies that $|w(\sigma)| = 1$ for some σ, $0 \leq \sigma \leq T$ and (48) implies that

$$w_k(t) \geq 0 \qquad \text{for} \quad 0 \leq t \leq T \tag{56}$$

These contradict to the conclusion of Theorem 1, and hence the first part of our assertion is proved. The second part is proved similarly. In fact, since the identically zero solution of (1) is stable, we may assume that $x(t)$ is sufficiently small for all $t > 0$ when $x(0)$ is sufficiently small. Therefore the argument of our first holds for $[nT, (n+1)T]$ for any positive number n, and hence each component $x_k(t)$ changes its sign on $[nT, (n+1)T]$, which shows the second part of our assertion. Thus the proof of Theorem 2 is completed.

2.3 Proof of Theorem 3

We shall prove that identically zero solution of (9) is asymptotically stable. The linear part of (9) is the following

$$\begin{aligned} \ddot{x} + k\dot{x} + ax &= b\dot{y} \\ \ddot{y} + k\dot{y} + ay &= -b\dot{x} \end{aligned} \tag{57}$$

which has the characteristic equation such that

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0 \tag{58}$$

where

$$a_1 = 2k,\ a_2 = k^2 + b^2 + 2a,\ a_3 = 2ak,\ a_4 = a^2 \tag{59}$$

It follows from [Routh-Hurwitz problem, 2] that generally every root of (58) has a negative real part if and only if

$$a_1 > 0,\ a_2 > 0,\ a_3(a_1a_2 - a_3) > a_2^2 a_4 \tag{60}$$

which is equal to the following in the case of (59)

$$(4-a)k^4 + 2Kk^2 - a(4-a)(2a+b)^2 > 0 \tag{61}$$

Since (10) and (11) implies (60), every root of (58) with (59) has a negative real part. Therefore [Theorem 1.1, 1] guarantees that the identically zero solution of (9) is asymptotically stable, and hence the proof is completed.

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