Some Properties on Mean Curvatures of Codimension-One Taut Foliations

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Abstract

Given a codimension-one foliation $\mathcal{F}$ of a not necessarily closed manifold $M$. We show a relation between the changes of Riemannian metrics and the mean curvature functions, and derive some consequences when $\mathcal{F}$ is a taut foliation. A relation between these results and a characterization of admissible vector fields is also discussed.

1 Introduction

Let $\mathcal{F}$ be a foliation of any codimension of a compact manifold $M$ and $X$ be a vector field on $M$. Recently, P. Schweitzer and P. Walczak [10] provided some necessary and sufficient conditions for $X$ to become the mean curvature vector of $\mathcal{F}$ with respect to some Riemannian metric on a closed manifold $M$. In a previous paper [7], the author studied the same problem for codimension-one foliations $\mathcal{F}$, and gave a necessary and sufficient condition for $X$ to become the mean curvature vector of $\mathcal{F}$ with respect to some Riemannian metric on $M$, which resembles the conditions given in the papers of the author ([4], [5], [6]). However, as the conditions given in the above paper are complicated, further studies are needed on this problem. In this paper, we give a relation between the changes of Riemannian metrics and the mean curvature functions, and derive some consequences when $\mathcal{F}$ is a taut foliation. A relation between these results and a characterization of admissible vector fields is also discussed.

We shall give some definitions, preliminaries and the results in § 2, and shall prove them in § 3. Some remarks are given in § 4.

2 Preliminaries and results

In this paper, we work in the $C^\infty$-category. In what follows, we always assume that foliations are of

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codimension-one and transversely oriented, and that the ambient manifolds are connected, oriented and of dimension \( n + 1 \geq 2 \), unless otherwise stated (see [1], [12] for the generalities on foliations).

Let \( g \) be a Riemannian metric of \( M \). Then there is a unique vector field orthogonal to \( F \) whose direction coincides with the given transverse orientation. We denote this vector field by \( N \). Orientations of \( M \) and \( F \) are related as follows: Let \( \{ E_1, E_2, \ldots, E_n \} \) be an oriented local orthonormal frame of \( T_1 F \). Then the orientation of \( M \) coincides with the one given by \( \{ N, E_1, E_2, \ldots, E_n \} \).

We denote by \( h_\sigma(x) \) the mean curvature of a leaf \( L \) at \( x \) with respect to \( g \) and \( N \), that is,
\[
h_\sigma = \sum_{i=1}^n \langle \nabla_{E_i} N, E_i \rangle,
\]
where \( \langle , \rangle \) means \( g( , ) \), \( \nabla \) is the Riemannian connection of \( (M, g) \) and \( \{ E_1, E_2, \ldots, E_n \} \) is an oriented local orthonormal frame of \( T_1 F \). The vector field \( h_\sigma = h_\sigma N \) is called the \textit{mean curvature vector} of \( F \) with respect to \( g \). A smooth function \( f \) on \( M \) is called \textit{admissible} if \( f = - h_\sigma \) for some Riemannian metric \( g \) (cf. [4], [13]). A characterization of admissible functions is given in [6] (see also [4], [5], [13]). We also call a vector field \( X \) on \( M \) \textit{admissible} if \( X = h_\sigma N \) for some Riemannian metric \( g \). A characterization of admissible vector fields is given in [7]. Define an \( n \)-form \( \chi_F \) on \( M \) by
\[
\chi_F(V_1, \ldots, V_n) = \det(\langle E_i, V_j \rangle)_{i,j=1,\ldots,n} \quad \text{for } V_j \in TM.
\]
The restriction \( \chi_F | L \) is the volume element of \( \langle L, L | g \rangle \) for \( L \in F \). Note that if \( \omega \) is the dual 1-form of \( N \), that is, \( \omega(V) = g(N, V) \) for \( V \in TM \), then \( dV_j = \omega \wedge \chi_F \), where \( dV_j \) is the volume element of \( (M, g) \).

The following Rummler’s result plays a key role in this paper.

**Proposition R** (Rummler [8]). \( d \chi_F = - h_\sigma dV_g = \text{div}_g(N) dV_g \), where \( \text{div}_g(N) \) is the divergence of \( N \) with respect to \( g \), that is, \( \text{div}_g(N) = \sum_{i=1}^n \langle \nabla_{E_i} N, E_i \rangle \).

A codimension-one foliation \( F \) is called \textit{taut} if there is a Riemannian metric \( g \) of \( M \) so that every leaf of \( F \) is a minimal submanifold of \( (M, g) \). A topological characterization of taut foliations of closed manifolds is given by Sullivan [11].

Our results are the following.

**Theorem 1.** Let \( (M, F) \) be a codimension-one taut foliation, and \( g \) be a Riemannian metric of \( M \) so that \( F \) is minimal, and \( N \) be the unit vector field on \( M \) defined above. Then for a smooth function \( f \) on \( M \) the vector field \( f N \) is admissible if and only if \( f \) is of the form \( \sigma^2 N(\varphi) \) for some smooth functions \( \sigma > 0 \) and \( \varphi \) on \( M \).

**Theorem 2.** Let \( (M, F) \) be a codimension-one foliation, and \( g \) be a Riemannian metric of \( M \). Let \( N \) be the unit vector field on \( M \) defined above. Then \( F \) is taut if and only if there are a positive smooth function \( \varphi \) and a vector field \( F \) tangent to \( F \) so that \( \text{div}_g(\varphi N + F) = 0 \).
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These results are local in nature, and hold for not necessarily closed manifold. In §4, we discuss these results from the viewpoint of the setting of Sullivan.

3 Proof of Theorems

Firstly, we prove a proposition, which is concerned with a relation between mean curvature functions and Riemannian metrics (cf. Lemma 3 in [3]).

**Proposition.** Let $\mathcal{F}$ be a codimension-one foliation of a Riemannian manifold $(M, g)$, $N$ be the unit vector field orthogonal to $\mathcal{F}$ defined as in Section 2, and $h$ be the mean curvature function of $\mathcal{F}$ with respect to $g$. Let $\tilde{g}$ be another Riemannian metric of $M$ and $\tilde{N}$ be the unit vector field orthogonal to $\mathcal{F}$ with respect to $\tilde{g}$. Set $\tilde{N} = \sigma N + F$ for a positive smooth function $\sigma$ on $M$ and $F \in \Gamma(\mathcal{F})$. Further, also set $\chi_\mathcal{F}|_\mathcal{F} = \varphi \chi_\mathcal{F}|_\mathcal{F}$ for a positive smooth function $\varphi$ on $M$. Then, for the mean curvature $\tilde{h}$ of $\mathcal{F}$ with respect to $\tilde{g}$, we have

$$
\tilde{h} = \sigma h - \sigma N(\log \varphi) - F(\log \frac{\varphi}{\sigma}) - \text{div}_g(F).
$$

(Proof.) Hereafter, we denote $\chi_\mathcal{F}$ and $\chi_\mathcal{F}$ by $\chi$ and $\chi$, respectively. Denote also $dV_{\tilde{g}}$ by $dV$ and $dV_{\tilde{g}}$ by $dV$, respectively. As $\tilde{h}$ does not depend on $\varphi_\mathcal{F}$ but only on $\chi$, we may assume that the metrics $\varphi_\mathcal{F}$ and $\varphi_\mathcal{F}$ satisfy the following relation as $\chi_\mathcal{F}|_\mathcal{F} = \varphi \chi_\mathcal{F}|_\mathcal{F}$: If $\{E_1, E_2, \cdots, E_n\}$ is a local orthonormal frame of $T \mathcal{F}$ with respect to $g$, then $\{E_1/\varphi, E_2, \cdots, E_n\}$ is a local orthonormal frame of $T \mathcal{F}$ with respect to $\tilde{g}$. We denote this frame by $\{\tilde{E}_1, \tilde{E}_2, \cdots, \tilde{E}_n\}$. Let $\omega$, $\omega_1$, $\omega_2$, $\cdots$, $\omega_n$ be the dual 1-forms of $N, E_1, E_2, \cdots, E_n$. Then it follows that

$$
\omega = \frac{1}{\sigma} \omega_1 = \frac{\varphi}{\sigma} \omega_1(F) \omega, \quad \omega_i = \omega_i - \frac{1}{\sigma} \omega_i(F) \omega \quad (i \geq 2).
$$

In fact, as $1 = \omega(N) = \omega(\sigma N + F) = \sigma \omega(N)$ and $\text{Ker} \omega = \text{Ker} \omega$, we have $\sigma \omega = \omega$. As $0 = \omega_1(N) = \omega_1(\sigma N + F) = \sigma \omega_1(N) + \omega_1(F)$, we have $\omega_1(N) = - (\varphi/\sigma) \omega_1(F)$. It follows that $\omega_1 = \frac{\varphi}{\sigma} \omega_1 - (\varphi/\sigma) \omega_1(F) \omega$. For $i \geq 2$, by the similar argument, we have $\omega_i = \omega_i - (\omega_i(F)/\sigma) \omega$. It follows that

$$
\omega = \omega \wedge \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n
$$

$$
= (\omega/\sigma) \wedge (\varphi \omega_1 - (\varphi_1(F)/\sigma) \omega) \wedge (\omega_2 - (\omega_2(F)/\sigma) \omega) \wedge \cdots \wedge (\omega_n - (\omega_n(F)/\sigma) \omega)
$$

$$
= (\varphi/\sigma) \omega \wedge \omega_1 \wedge \cdots \wedge \omega_n
$$

$$
= \frac{\varphi}{\sigma} dV.
$$
We also have
\[
\chi = \varphi n \wedge \varphi n \wedge \cdots \wedge \varphi n
\]
\[
= \varphi (\varphi n - (\varphi n(F) / \sigma) \omega) \wedge (\varphi n - (\varphi n(F) / \sigma) \omega) \wedge \cdots \wedge (\varphi n - (\varphi n(F) / \sigma) \omega)
\]
\[
= \varphi \varphi n \wedge \cdots \wedge \varphi n - \varphi \sum_{i=1}^{n} ((\varphi n(F) / \sigma) \omega) \wedge \varphi n \wedge \cdots \wedge \varphi n \wedge \varphi n \wedge \cdots \wedge \varphi n
\]
\[
= \varphi \chi + \frac{\varphi n}{\sigma} \omega \wedge \left( \sum_{i=1}^{n} (-1)^{i} \varphi n(F) \varphi n \wedge \cdots \wedge \varphi n \wedge \varphi n \wedge \cdots \wedge \varphi n \right)
\]
\[
= \varphi \chi + \frac{\varphi n}{\sigma} F dV,
\]
where \( \ell_{F} \) denotes the interior product by \( F \).

Now we are in a position to prove our assertion. As, by Proposition R, \( d \chi = -h dV \) and \( d \chi = -\bar{h} \bar{d}V \), we have
\[
-\bar{h} \bar{d}V = d \chi = d \left( \varphi \chi + \frac{\varphi n}{\sigma} F dV \right)
\]
\[
= d \varphi \wedge \chi + \varphi d \chi + d \left( \frac{\varphi n}{\sigma} F dV \right) \wedge \ell_{F} dV + \frac{\varphi n}{\sigma} d \ell_{F} dV
\]
\[
= \left( b(N) - \varphi h + F \left( \frac{\varphi n}{\sigma} F \right) \right) dV
\]
\[
= \left( b(N) - \varphi h + F \left( \frac{\varphi n}{\sigma} F \right) \right) \frac{\varphi n}{\sigma} dV.
\]
Thus, we have
\[
\bar{h} = \sigma h - \sigma N (\log \varphi) - F (\log \frac{\varphi n}{\sigma} F) - \operatorname{div}_{g}(F).
\]

(Proof of Theorem 1.) Firstly note that, by Proposition, we have the following.

Assertion. Let \( F \) be a codimension-one foliation of a Riemannian manifold \((M, g)\), \( N \) be the unit vector field orthogonal to \( F \), and \( h \) be the mean curvature function of \( F \) with respect to \( g \). If \( \bar{g} \) is another Riemannian metric of \( M \) so that \( F \perp N, \bar{N} \), and \( H = f \bar{N} \), then \( f = \sigma^{2}(h - N(\varphi)) \) for some smooth functions \( \sigma > 0 \) and \( \varphi \) on \( M \).

Indeed, in Proposition, if we set \( F = 0, N = \sigma N, \) \( \bar{N} = \varphi \bar{N} \), then we get \( \bar{h} = \sigma h - \sigma N (\log \varphi) \).

As \( \bar{H} = \bar{h} \bar{N} = \bar{h} \sigma N = fN, \) it follows that \( f = \sigma^{2}(h - N(\log \varphi)). \)

Assume that \( F \) is minimal with respect to \( g \). Then, we have \( h = 0. \) By the assertion, \( f \) is of the form \( \sigma^{2}N(\varphi) \) for some smooth functions \( \sigma > 0 \) and \( \varphi \) on \( M \).

Conversely, assume that \( f \) is of the form \( \sigma^{2}N(\varphi) \) for some smooth functions \( \sigma > 0 \) and \( \varphi \) on \( M \). If we choose a Riemannian metric \( \bar{g} \) of \( M \) so that \( F \perp N, \bar{N} = \sigma N, \) \( \bar{N} = \varphi \bar{N} \), then, as \( h = 0, \) from the proof of the assertion, we have the desired result. This completes the proof.

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(Proof of Theorem 2.) We shall use the same notations as in Proposition. Let \( g \) be any Riemannian metric of \( M \). Assume that there are a positive smooth function \( \varphi \) and a vector field \( F \) tangent to \( \mathcal{F} \) so that \( \text{div}_g(\varphi N + F) = 0 \). Choose a Riemannian metric \( \overline{g} \) with \( \mathcal{F} \perp N + (1/\varphi)F \), \( \overline{N} = N + (1/\varphi)F \), and \( \overline{x} = \varphi \chi \mid \mathcal{F} \). Then, by Proposition, we have \( \overline{h} = h - N(\log \varphi) - (1/\varphi)F(\log \varphi) - \text{div}_g((1/\varphi)F) \), because \( \alpha \equiv 1 \) on \( M \). As \( h = N(\log \varphi) - (1/\varphi)F(\log \varphi) - \text{div}_g((1/\varphi)F) = - (1/\varphi)(\text{div}_g(\varphi N + F)) = 0 \), by assumption, we have \( \overline{h} = 0 \), which shows that \( \mathcal{F} \) is taut.

Conversely, assume that \( \mathcal{F} \) is minimal with respect to some Riemannian metric \( g \) of \( M \). We show that there are a positive smooth function \( \varphi \) and a vector field \( F \) tangent to \( \mathcal{F} \) so that \( \text{div}_g(\varphi N + F) = 0 \). Let \( \overline{N} = \alpha N + Z \), where \( Z \in \Gamma(\mathcal{F}^0) \), be the unit vector field orthogonal to \( \mathcal{F} \) with respect to \( \overline{g} \), and \( \varphi \) be a smooth function satisfying \( \overline{x} = \varphi \chi \mid \mathcal{F} \). Then, from the proof of Proposition, we have

\[
0 = N(\varphi) - \varphi h + Z \left( \frac{\varphi}{\sigma} \right) + \frac{\varphi}{\sigma} \text{div}_g(Z) = \text{div}_g(\varphi N + \varphi Z).
\]

By setting \( F = (\varphi \alpha Z) \), we have the desired result.

As corollaries to Theorem 2, we have

**Corollary 1.** Let \( (M, \mathcal{F}) \) be a codimension-one foliation, and \( g \) be a Riemannian metric of \( M \). Let \( N \) be the unit vector field on \( M \) defined as above. Then there is a Riemannian metric \( \overline{g} \) that makes \( \mathcal{F} \) minimal with \( \overline{x} = \chi \mid \mathcal{F} \) if and only if there is a vector field \( F \) tangent to \( \mathcal{F} \) so that \( \text{div}_g(N + F) = 0 \).

**Corollary 2.** Let \( (M, \mathcal{F}) \) be a codimension-one foliation and \( X \) be a non-vanishing divergent-free vector field, that is, \( \text{div}X = 0 \) on \( M \). Then, any codimension-one foliation transverse to \( X \) is taut.

### 4 Concluding remarks

In this section, we give some remarks on a relation between the results of this paper and the conditions given in [7]. In order to recall the characterization of admissible vector fields given in [7], firstly recall the set-up by Sullivan [11]. In what follows, we assume that \( M \) is a closed oriented manifold. Let \( D_p \) be the space of \( p \)-currents, and \( D^p \) be the space of differential \( p \)-forms on \( M \) with the \( C^\infty \) topology. It is well known that \( D^p \) is the dual space of \( D_p \) (cf. Schwartz [9]). Let \( x \in M \) and \( \{e_1, \ldots, e_n\} \) be an oriented basis of \( T_x \mathcal{F} \). We define the Dirac current \( \delta_{e_1 \wedge \cdots \wedge e_n} \) by

\[
\delta_{e_1 \wedge \cdots \wedge e_n}(\phi) = \phi_x(e_1 \wedge \cdots \wedge e_n) \quad \text{for} \quad \phi \in D^n.
\]

and set \( C^\mathcal{F} \) to be the closed convex cone in \( D_n \) spanned by Dirac currents \( \delta_{e_1 \wedge \cdots \wedge e_n} \) for all oriented bases \( \{e_1, \ldots, e_n\} \) of \( T_x \mathcal{F} \) and \( x \in M \). We denote a base of \( C^\mathcal{F} \) by \( \mathcal{C} \), which is an inverse image \( L^{-1}(1) \) of a suitable continuous linear functional \( L : D_n \to \mathbb{R} \). It is known that the base \( \mathcal{C} \) is compact if \( L \) is suitably chosen. In the following, we assume that \( \mathcal{C} \) is compact.
Let $X$ be a vector field on $M$. Define the closed linear subspace $P(X)$ of $D_n$ generated by all the Dirac currents $\delta_{\mathcal{X}(x)} \wedge v_1 \wedge \cdots \wedge v_{n-1}$ with $v_1, \ldots, v_{n-1} \in T_x \mathcal{F}$ and $x \in M$ (see [10] for more details), where

$$
\delta_{\mathcal{X}(x)} \wedge v_1 \wedge \cdots \wedge v_{n-1}(\phi) = \phi_\mathcal{X}(X(x) \wedge v_1 \wedge \cdots \wedge v_{n-1}) \quad \text{for} \quad \phi \in D^n.
$$

Let $\partial : C_{n+1} \to C_n$ be the boundary operator and set $B = \partial (C_{n+1})$. In these settings, we gave the following characterization of admissible vector fields on a closed manifold $M$ (Theorem 2 in [7]):

For a vector field $X$ on $M$, the following two conditions are equivalent.

(1) $X$ is admissible.

(2) There are a volume element $dV$, a non-vanishing vector field $Z$ transverse to whose direction coincides with the given transverse orientation of $\mathcal{F}$, a smooth function $f$ on $M$, and a neighborhood $U$ of $0 \in D_n$ such that

(i) $X = -fZ$,

(ii) $\int_M f dV = 0$,

(iii) $\int_c f dV = 0$ for all $c \in \partial^{-1}(P(X) \cap B)$, and

(iv) $\inf \{ \int_c f dV \mid c \in \partial^{-1}((C + P(X) + U) \cap B) \} > 0$.

Concerning Theorem 1, we show an implication: If $f$ is of the form $\sigma^2 N(\phi)$, then $fN$ is admissible.

Note that if $\mathcal{F}$ is taut, then it is easy to see that $(C + P(X) + U) \cap B = \emptyset$. Thus the condition (iv) becomes void. Set $\int dV = (1/\sigma^2) dV$. Then, as $f dV = N(\phi) dV = d(\phi \chi)$, because $d \chi = 0$, it follows that

$$
\int_M f dV = \int_M d(\phi \chi) = \int_M \phi \chi = 0,
$$

because $\chi \mid_{P(N)} = 0$ and $\partial c \in P(N)$, which means the condition (iii) is satisfied.

Concerning Theorem 2, we show an implication: If $\text{div}_\mathcal{F}(\phi N + F) = 0$, then $\mathcal{F}$ is taut.

Set $\psi = \iota_{(\phi N + F)} dV$. Then, $d \psi = d\iota_{(\phi N + F)} dV = L_{(\phi N + F)} dV = \text{div}_\mathcal{F}(\phi N + F) = 0$. Further, as $\psi \mid_\mathcal{F} > 0$ and $\psi \mid_{(\phi N + F)} = 0$, it is easy to see that the vector field $0 \cdot N = 0$ is admissible, that is, $\mathcal{F}$ is taut.

References


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