Ann. Rep. Fac. Edu., Iwate Univ., Vol.69 (Feb. 2010), 103 ~ 109

Some Properties on Mean Curvatures of Codimension-One Taut Foliations

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(Received Dec. 9, 2009)

Abstract

Given a codimension-one foliation $\mathcal{F}$ of a not necessarily closed manifold M. We show a relation between the changes of Riemannian metrics and the mean curvature functions, and derive some consequences when $\mathcal{F}$ is a taut foliation. A relation between these results and a characterization of admissible vector fields is also discussed.

1 Introduction

Let $\mathcal{F}$ be a foliation of any codimension of a compact manifold M and X be a vector field on M. Recently, P. Schweitzer and P. Walczak [10] provided some necessary and sufficient conditions for X to become the mean curvature vector of $\mathcal{F}$ with respect to some Riemannian metric on a closed manifold M. In a previous paper [7], the author studied the same problem for codimension-one foliations $\mathcal{F}$, and gave a necessary and sufficient condition for X to become the mean curvature vector of $\mathcal{F}$ with respect to some Riemannian metric on M, which resembles the conditions given in the papers of the author ([4], [5], [6]). However, as the conditions given in the above paper are complicated, further studies are needed on this problem. In this paper, we give a relation between the changes of Riemannian metrics and the mean curvature functions, and derive some consequences when $\mathcal{F}$ is a taut foliation. A relation between these results and a characterization of admissible vector fields is also discussed.

We shall give some definitions, preliminaries and the results in §2, and shall prove them in §3. Some remarks are given in §4.

2 Preliminaries and results

In this paper, we work in the C^∞-category. In what follows, we always assume that foliations are of

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codimension-one and transversely oriented, and that the ambient manifolds are connected, oriented and of dimension $n+1\geq 2$, unless otherwise stated (see [1], [12] for the generalities on foliations).

Let g be a Riemannian metric of M. Then there is a unique vector field orthogonal to $\mathcal{F}$ whose direction coincides with the given transverse orientation. We denote this vector field by N. Orientations of M and $\mathcal{F}$ are related as follows: Let $\{E_1, E_2, \ldots, E_n\}$ be an oriented local orthonormal frame of $T\mathcal{F}$. Then the orientation of M coincides with the one given by $\{N, E_1, E_2, \ldots, E_n\}$.

We denote by $h_g(x)$ the mean curvature of a leaf L at x with respect to g and N, that is,

$$h_g = \sum_{i=1}^{n} \langle \nabla_{E_i} E_i, N \rangle,$$

where $\langle , \rangle$ means $g(,)$, ∇ is the Riemannian connection of (M, g) and $\{E_1, E_2, \ldots, E_n\}$ is an oriented local orthonormal frame of $T\mathcal{F}$. The vector field $H_g = h_g N$ is called the *mean curvature vector* of $\mathcal{F}$ with respect to g. A smooth function f on M is called *admissible* if $f = -h_g$ for some Riemannian metric g (cf. [4], [13]). A characterization of admissible functions is given in [6] (see also [4], [5], [13]). We also call a vector field X on M *admissible* if $X = Hg$ for some Riemannian metric g. A characterization of admissible vector fields is given in [7]. Define an n-form $\chi_{\mathcal{F}}$ on M by

$$\chi_{\mathcal{F}}(V_1, \ldots, V_n) = \det(\langle E_i, V_j \rangle)_{i,j=1,\ldots,n} \quad \text{for} \quad V_j \in TM.$$

The restriction $\chi_{\mathcal{F}}|L$ is the volume element of $(L, L|g)$ for $L \in \mathcal{F}$. Note that if ω is the dual 1-form of N, that is, $\omega(V) = g(N, V)$ for $V \in TM$, then $dV_g = \omega \wedge \chi_{\mathcal{F}}$, where dV_g is the volume element of (M, g). The following Rummler's result plays a key role in this paper.

Proposition R (Rummler [8]). $d\chi_{\mathcal{F}} = -h_g dV_g = \mathrm{div}_g(N) dV_g$, where $\mathrm{div}_g(N)$ is the divergence of N with respect to g, that is, $\mathrm{div}_g(N) = \sum_{i=1}^{n} \langle \nabla_{E_i} N, E_i \rangle$.

A codimension-one foliation $\mathcal{F}$ is called *taut* if there is a Riemannian metric g of M so that every leaf of $\mathcal{F}$ is a minimal submanifold of (M, g). A topological characterization of taut foliations of closed manifolds is given by Sullivan [11].

Our results are the following.

Theorem 1. Let $(M, \mathcal{F})$ be a codimension-one taut foliation, and g be a Riemannian metric of M so that $\mathcal{F}$ is minimal, and N be the unit vector field on M defined above. Then for a smooth function f on M the vector field fN is admissible if and only if f is of the form $\sigma^2 N(\varphi)$ for some smooth functions $\sigma > 0$ and φ on M.

Theorem 2. Let $(M, \mathcal{F})$ be a codimension-one foliation, and g be a Riemannian metric of M. Let N be the unit vector field on M defined above. Then $\mathcal{F}$ is taut if and only if there are a positive smooth function φ and a vector field F tangent to $\mathcal{F}$ so that $\mathrm{div}_g(\varphi N + F) = 0$.

These results are local in nature, and hold for not necessarily closed manifold. In § 4, we discuss these results from the view point of the setting of Sullivan.

3 Proof of Theorems

Firstly, we prove a proposition, which is concerned with a relation between mean curvature functions and Riemannian metrics (cf. Lemma 3 in [3]).

Proposition. Let $\mathcal{F}$ be a codimension-one foliation of a Riemannian manifold (M, g), N be the unit vector field orthogonal to $\mathcal{F}$ defined as in Section 2, and h be the mean curvature function of $\mathcal{F}$ with respect to g. Let $\bar{g}$ be another Riemannian metric of M and $\overline{N}$ be the unit vector field orthogonal to $\mathcal{F}$ with respect to $\bar{g}$. Set $\overline{N} = \sigma N + F$ for a positive smooth function σ on M and $F \in \Gamma(\mathcal{F})$. Further, also set $\overline{\chi_{\mathcal{F}}}|_{\mathcal{F}} = \varphi \chi_{\mathcal{F}}|_{\mathcal{F}}$ for a positive smooth function φ on M. Then, for the mean curvature $\bar{h}$ of $\mathcal{F}$ with respect to $\bar{g}$, we have

$$\bar{h} = \sigma h - \sigma N(\log \varphi) - F(\log \frac{\varphi}{\sigma}) - \mathrm{div}_g(F).$$

(Proof.) Hereafter, we denote $\chi_{\mathcal{F}}$ and $\overline{\chi_{\mathcal{F}}}$ by χ and $\overline{\chi}$, respectively. Denote also dV_g by dV and $dV_{\bar{g}}$ by $\overline{dV}$, respectively. As $\bar{h}$ does not depend on $g|_{\mathcal{F}}$ but only on χ, we may assume that the metrics $g|_{\mathcal{F}}$ and $\bar{g}|_{\mathcal{F}}$ satisfy the following relation as $\overline{\chi}|_{\mathcal{F}} = \varphi \overline{\chi}|_{\mathcal{F}}$: If $\{E_1, E_2, \cdots, E_n\}$ is a local orthonormal frame of $T\mathcal{F}$ with respect to g, then $\{E_1/\varphi, E_2, \cdots, E_n\}$ is a local orthonormal frame of $T\mathcal{F}$ with respect to $\bar{g}$. We denote this frame by $\{\overline{E}_1, \overline{E}_2, \cdots, \overline{E}_n\}$. Let $\overline{\omega}, \overline{\omega}_1, \overline{\omega}_2, \cdots, \overline{\omega}_n$ be the dual 1-forms of $\overline{N}, \overline{E}_1, \overline{E}_2, \cdots, \overline{E}_n$. Then it follows that

$$\overline{\omega} = \frac{1}{\sigma}\omega, \ \overline{\omega}_1 = \varphi\omega_1 - \frac{\varphi}{\sigma}\omega_1(F)\omega, \ \overline{\omega}_i = \omega_i - \frac{1}{\sigma}\omega_i(F)\omega \quad (i \geq 2).$$

In fact, as $1 = \overline{\omega}(\overline{N}) = \overline{\omega}(\sigma N + F) = \sigma\overline{\omega}(N)$ and Ker ω = Ker $\overline{\omega}$, we have $\sigma\overline{\omega} = \omega$. As $0 = \overline{\omega}_1(\overline{N}) = \overline{\omega}_1(\sigma N + F) = \sigma\,\overline{\omega}_1(N) + \overline{\omega}_1(F)$, we have $\overline{\omega}_1(N) = -(\varphi/\sigma)\,\omega_1(F)$. It follows that $\overline{\omega}_1 = \varphi\omega_1 - (\varphi/\sigma)\omega_1(F)\omega$. For $i \geq 2$, by the similar argument, we have $\overline{\omega}_i = \omega_i - (\omega_i(F)/\sigma)\,\omega$. It follows that

$$\begin{aligned}
\overline{dV} &= \overline{\omega} \wedge \overline{\omega}_1 \wedge \overline{\omega}_2 \wedge \cdots \wedge \overline{\omega}_n \\
&= (\omega/\sigma) \wedge (\varphi\omega_1 - (\varphi\omega_1(F)/\sigma)\omega) \wedge (\omega_2 - (\omega_2(F)/\sigma)\omega) \wedge \cdots \wedge (\omega_n - (\omega_n(F)/\sigma)\omega) \\
&= (\varphi/\sigma)\omega \wedge \omega_1 \wedge \cdots \wedge \omega_n \\
&= \frac{\varphi}{\sigma} dV.
\end{aligned}$$

We also have

$$\begin{aligned}
\overline{\chi} &= \overline{\omega}_1 \wedge \overline{\omega}_2 \wedge \cdots \wedge \overline{\omega}_n \\
&= \varphi(\omega_1 - (\omega_1(F)/\sigma)\omega) \wedge (\omega_2 - (\omega_2(F)/\sigma)\omega) \wedge \cdots \wedge (\omega_n - (\omega_n(F)/\sigma)\omega) \\
&= \varphi\omega_1 \wedge \cdots \wedge \omega_n - \varphi \sum_{i=1}^{n} ((\omega_i(F)/\sigma)\omega) \wedge \omega_1 \wedge \cdots \wedge \omega_{i-1} \wedge \omega_{i+1} \wedge \cdots \wedge \omega_n \\
&= \varphi\chi + \frac{\varphi}{\sigma}\omega \wedge \left(\sum_{i=1}^{n} (-1)^{-i} \omega_i(F)\omega_1 \wedge \cdots \wedge \omega_{i-1} \wedge \omega_{i+1} \wedge \cdots \wedge \omega_n \right) \\
&= \varphi\chi + \frac{\varphi}{\sigma}\iota_F dV,
\end{aligned}$$

where ι_F denotes the interior product by F.

Now we are in a position to prove our assertion. As, by Proposition R, $d\chi = -h\,dV$ and $d\overline{\chi} = -\overline{h}\,\overline{dV}$, we have

$$\begin{aligned}
-\overline{h}\,\overline{dV} &= d\overline{\chi} = d\left(\varphi\chi + \frac{\varphi}{\sigma}\iota_F dV\right) \\
&= d\varphi \wedge \chi + \varphi d\chi + d\left(\frac{\varphi}{\sigma}\right) \wedge \iota_F dV + \frac{\varphi}{\sigma} d\iota_F dV \\
&= \left(N(\varphi) - \varphi h + F\left(\frac{\varphi}{\sigma}\right) + \frac{\varphi}{\sigma}\mathrm{div}_g(F)\right) dV \\
&= \left(N(\varphi) - \varphi h + F\left(\frac{\varphi}{\sigma}\right) + \frac{\varphi}{\sigma}\mathrm{div}_g(F)\right) \frac{\sigma}{\varphi}\overline{dV}.
\end{aligned}$$

Thus, we have

$$\overline{h} = \sigma h - \sigma N(\log\varphi) - F(\log\frac{\varphi}{\sigma}) - \mathrm{div}_g(F).$$

(Proof of Theorem 1.) Firstly note that, by Proposition, we have the following.

Assertion. Let $\mathcal{F}$ be a codimension-one foliation of a Riemannian manifold (M, g), N be the unit vector field orthogonal to $\mathcal{F}$, and h be the mean curvature function of $\mathcal{F}$ with respect to g. If $\overline{g}$ is another Riemannian metric of M so that $\mathcal{F} \perp N, \overline{g}(\overline{N},\overline{N}) = 1$, and $\overline{H} = fN$, then $f = \sigma^2(h - N(\varphi))$ for some smooth functions $\sigma > 0$ and φ on M.

Indeed, in Proposition, if we set $F = 0$, $\overline{N} = \sigma N$, and $\overline{\chi} = \varphi\chi$, then we get

$$\overline{h} = \sigma h - \sigma N(\log\varphi).$$

As $\overline{H} = \overline{h}\,\overline{N} = \overline{h}\sigma N = fN$, it follows that $f = \sigma^2(h - N(\log\varphi))$.

Assume that $\mathcal{F}$ is minimal with respect to g. Then, we have $h = 0$. By the assertion, f is of the form $\sigma^2 N(\varphi)$ for some smooth functions $\sigma > 0$ and φ on M.

Conversely, assume that f is of the form $\sigma^2 N(\varphi)$for some smooth functions $\sigma > 0$ and φ on M. If we choose a Riemannian metric $\overline{g}$ of M so that $\mathcal{F} \perp N$, $\overline{N} = \sigma$N, and $\overline{\chi} = e^{-\varphi}\chi$, then, as $h = 0$, from the proof of the assertion, we have the desired result. This completes the proof.

(*Proof of Theorem 2.*) We shall use the same notations as in Proposition. Let g be any Riemannian metric of M. Assume that there are a positive smooth function φ and a vector field F tangent to $\mathcal{F}$ so that $\mathrm{div}_g(\varphi N + F) = 0$. Choose a Riemannian metric $\bar{g}$ with $\mathcal{F} \perp N + (1/\varphi)F$, $\overline{N} = N + (1/\varphi)F$, and $\overline{\chi}|_{\mathcal{F}} = \varphi\chi|_{\mathcal{F}}$. Then, by Proposition, we have $\overline{h} = h - N(\log\varphi) - (1/\varphi)F(\log\varphi) - \mathrm{div}_g((1/\varphi)F)$, because $\sigma \equiv 1$ on M. As $h - N(\log\varphi) - (1/\varphi)F(\log\varphi) - \mathrm{div}_g((1/\varphi)F) = -(1/\varphi)(\mathrm{div}_g(\varphi N + F)) = 0$, by assumption, we have $\overline{h} = 0$, which shows that $\mathcal{F}$ is taut.

Conversely, assume that $\mathcal{F}$ is minimal with respect to some Riemannian metric $\bar{g}$ of M. We show that there are a positive smooth function φ and a vector field F tangent to $\mathcal{F}$ so that $\mathrm{div}_g(\varphi N + F) = 0$. Let $\overline{N} = \sigma N + Z$, where $Z \in \Gamma(\mathcal{F})$, be the unit vector field orthogonal to $\mathcal{F}$ with respect to $\bar{g}$, and φ be a smooth function satisfying $\overline{\chi}|_{\mathcal{F}} = \varphi\chi|_{\mathcal{F}}$. Then, from the proof of Proposition, we have

$$0 = N(\varphi) - \varphi h + Z\left(\frac{\varphi}{\sigma}\right) + \frac{\varphi}{\sigma}\mathrm{div}_g(Z) = \mathrm{div}_g(\varphi N + \frac{\varphi}{\sigma}Z).$$

By setting $F = (\varphi/\sigma)Z$, we have the desired result.

As corollaries to Theorem 2, we have

Corollary 1. Let $(M, \mathcal{F})$ be a codimension-one foliation, and g be a Riemannian metric of M. Let N be the unit vector field on M defined as above. Then there is a Riemannian metric $\bar{g}$ that makes $\mathcal{F}$ minimal with $\overline{\chi}|_{\mathcal{F}} = \chi|_{\mathcal{F}}$ if and only if there is a vector field F tangent to $\mathcal{F}$ so that $\mathrm{div}_g(N + F) = 0$.

Corollary 2. Let $(M, \mathcal{F})$ be a codimension-one foliation and X be a non-vanishing divergent-free vector field, that is, $\mathrm{div}X = 0$ on M. Then, any codimension-one foliation transverse to X is taut.

4 Concluding remarks

In this section, we give some remarks on a relation between the results of this paper and the conditions given in [7]. In order to recall the characterization of admissible vector fields given in [7], firstly recall the set-up by Sullivan [11]. In what follows, we assume that M is a closed oriented manifold. Let D_p be the space of p-currents, and D^p be the space of differential p-forms on M with the C^∞ topology. It is well known that D^p is the dual space of D_p (cf. Schwartz [9]). Let $x \in M$ and $\{e_1, \dots, e_n\}$ be an oriented basis of $T_x\mathcal{F}$. We define the Dirac current $\delta_{e_1\wedge\cdots\wedge e_n}$ by

$$\delta_{e_1\wedge\cdots\wedge e_n}(\phi) = \phi_x(e_1 \wedge \cdots \wedge e_n) \quad \text{for} \quad \phi \in D^n,$$

and set $C_{\mathcal{F}}$ to be the closed convex cone in D_n spanned by Dirac currents $\delta_{e_1\wedge\cdots\wedge e_n}$ for all oriented bases $\{e_1, \dots, e_n\}$ of $T_x\mathcal{F}$ and $x \in M$. We denote a base of $C_{\mathcal{F}}$ by $\mathbf{C}$, which is an inverse image $L^{-1}(1)$ of a suitable continuous linear functional $L : D_n \to \mathbf{R}$. It is known that the base $\mathbf{C}$ is compact if L is suitably chosen. In the following, we assume that $\mathbf{C}$ is compact.

Let X be a vector field on M. Define the closed linear subspace $P(X)$ of D_n generated by all the Dirac currents $\delta_{X(x)\wedge v_1\wedge\cdots\wedge v_{n-1}}$ with $v_1, \ldots, v_{n-1} \in T_x\mathcal{F}$ and $x \in M$ (see [10] for more details), where $\delta_{X(x)\wedge v_1\wedge\cdots\wedge v_{n-1}}$ is defined by

$$\delta_{X(x)\wedge v_1\wedge\cdots\wedge v_{n-1}}(\phi) = \phi_x(X(x)\wedge v_1\wedge\cdots\wedge v_{n-1}) \quad \text{for} \quad \phi \in D^n.$$

Let $\partial : C_{n+1} \to C_n$ be the boundary operator and set $B = \partial(C_{n+1})$. In these settings, we gave the following characterization of admissible vector fields on a closed manifold M (Theorem 2 in [7]):

For a vector field X on M, the following two conditions are equivalent.

(1) X is admissible.

(2) There are a volume element dV, a non-vanishing vector field Z transverse to $\mathcal{F}$ whose direction coincides with the given transverse orientation of $\mathcal{F}$, a smooth function f on M, and a neighborhood U of $0 \in D_n$ such that

(i) $X = -fZ$,

(ii) $\int_M f dV = 0$,

(iii) $\int_c f dV = 0$ for all $c \in \partial^{-1}(P(X)\cap B)$, and

(iv) $\inf\{\int_c f dV \mid c \in \partial^{-1}((\mathbf{C} + P(X) + U)\cap B)\} > 0$.

Concerning Theorem 1, we show an implication: If f is of the form $\sigma^2 N(\varphi)$, then fN is admissible.

Note that if $\mathcal{F}$ is taut, then it is easy to see that $(\mathrm{C} + P(X) + U) \cap B = \emptyset$. Thus the condition (iv) becomes void. Set $\overline{dV} = (1/\sigma^2)dV$. Then, as $f\overline{dV} = N(\varphi)dV = d(\varphi\chi)$, because $d\chi = 0$, it follows that $\int_M f\overline{dV} = \int_M d(\varphi\chi) = 0$, which means that the condition (ii) is satisfied. $\int_c f\overline{dV} = \int_c d(\varphi\chi) = \int_{\partial c}\varphi\chi = 0$, because $\chi|_{P(N)} = 0$ and $\partial c \in P(N)$, which means the condition (iii) is satisfied.

Concerning Theorem 2, we show an implication: If $\mathrm{div}_g(\varphi N + F) = 0$, then $\mathcal{F}$ is taut.

Set $\psi = \iota_{(\varphi N+F)}dV$. Then, $d\psi = d\iota_{(\varphi N+F)}dV = L_{(\varphi N+F)}dV = \mathrm{div}_g(\varphi N + F) = 0$. Further, as $\psi|_{\mathcal{F}} > 0$ and $\psi|_{P(\varphi N+F)} = 0$, it is easy to see that the vector field $0 \cdot N = 0$ is admissible, that is, $\mathcal{F}$ is taut.

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