On the Minimal Covering of 3-dimensional Hamming Scheme

Minoru Numata*
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1. Introduction

$R$, $S$, and $T$ are finite sets, and we put $R = \{1, 2, \ldots, r\}$, $S = \{1, 2, \ldots, s\}$, and $T = \{1, 2, \ldots, t\}$. Elements of $R$, $S$, and $T$ are called row numbers, column numbers, and vertical numbers, respectively.

A subset $F$ of the Cartesian product $R \times S \times T$ of three sets $R$, $S$, and $T$ is a covering of $R \times S \times T$ when $F$ satisfies the following condition; for each element $(i, j, k)$ of $R \times S \times T$, there is the element $(e, f, g)$ of $F$ such that $i-e$ and $j-f$, or $j-f$ and $k=g$, or $i=e$ and $k=g$. In other words $F$ is a covering of 3-dimensional Hamming scheme $R \times S \times T$.

A subset $J$ of $R \times S \times T$ is a 3-directed covering of $R \times S \times T$ when $J$ satisfies the following condition; for each element $(i, j, k)$ of $R \times S \times T$, there are $e, f,$ and $g$ such that $(e, j, k)$, $(i, f, k)$, and $(i, j, g)$ are the elements of $J$. The number of elements of $J$ is at least $\max\{rs, st, tr\}$.

Let $A$ be the set of the element $(i, j, k)$ of $R \times S \times T$ which satisfies the condition that $i+j+k \equiv 0 \mod (\min\{r, s, t\})$. Then $A$ is 3-directed covering of $R \times S \times T$, and the number of elements of $A$ is equal to $\max\{rs, st, tr\}$.

Let $A$ be the minimal 3-directed covering of $R \times S \times T$ when $A$ is 3-directed covering of $R \times S \times T$, and when the number of elements of $A$ is equal to $\max\{rs, st, tr\}$.

We shall call $A$ as minimal 3-directed covering of $R \times S \times T$ when $A$ is 3-directed covering of $R \times S \times T$, and when the number of elements of $A$ is equal to $\max\{rs, st, tr\}$.

Let $R$ be the disjoint union of subsets $R_1$ and $R_2$ of $R$, $S$ be the disjoint union of subsets $S_1$ and $S_2$ of $S$, $T$ be the disjoint union of subsets $T_1$ and $T_2$ of $T$. Let $I_1$ be a 3-directed covering of $R_1 \times S_1 \times T_1$, and $I_2$ be a 3-directed covering of $R_2 \times S_2 \times T_2$. Then the union of $I_1$ and $I_2$ is a covering of $R \times S \times T$.

Hereafter we assure $r \leq s \leq t$. We put

$$|R_1| = |S_1| = x, \quad |T_1| = t-r+x.$$

Let $I_1$ be the minimal 3-directed covering of $R_1 \times S_1 \times T_1$, and $I_2$ be the minimal 3-
directed coverings of $R_xS\times T$. Then the number of elements of the union of $I_1$ and $I_2$ is

$$x(t-r+x)+(r-x)(s-x)=2\left(x-\frac{2r+s-t}{4}\right)^2+rs-\frac{(2r+s-t)^2}{8}.$$  

If $2r+s-t\geq 0$, for integer $x$ the minimum of the above quadratic form is equal to

$$rs-\left\lfloor\frac{(2r+s-t)^2}{8}\right\rfloor.$$

Thus we have the following conjecture.

Conjecture. Let $I'$ be a covering of $R\times S\times T$ where $|R|=r$, $|S|=s$, and $|T|=t$; $r\leq s\leq t$. Then the number of elements of $I'$ is at least

$$rs-\left\lfloor\frac{(2r+s-t)^2}{8}\right\rfloor$$

if $2r+s\geq t+2$ and

$$rs-\left\lceil\frac{(2r+s-t)^2}{8}\right\rceil$$

if $2r+s<t+2$.

When $2r+s\geq t+2$ and the number of elements of $I'$ is equal to just

$$rs-\left\lfloor\frac{(2r+s-t)^2}{8}\right\rfloor,$$

then $I'$ is the union of two 3-directed coverings as the mentioned above, except few examples.

When $r=s$, we can prove this conjecture.

Theorem. Let $V$ and $T$ be finite sets, where $V=\{1,2,\ldots,v\}$ and $T=\{1,2,\ldots,t\}$, and $v\leq t\leq 3v-2$. Let $I'$ be a covering of $V\times V\times T$. Then

$$|I'|\geq v^2-\left\lfloor\frac{(3v-t)^2}{8}\right\rfloor.$$

When the equality holds, except the unique example of the case $v=4$ and $t=6$, and the examples of the case $t=v+2$, $I'$ is constructed as follows.

$I'$ is the union of $I_1$ and $I_2$ such that $I_1$ is a minimal 3-directed covering of $V_1\times V_1\times T_1$, and $I_2$ is a minimal 3-directed covering of $V_2\times V_2\times T_2$, where $V$ and $T$ are the disjoint union of $V_1$ and $V_2$, $T_1$ and $T_2$, respectively.

Furthermore $|V_1|=|T_1|=d$ where

$$d=v-j$$

if $3v-t=4j$

$$d=v-j$$

if $3v-t=4j+1$

$$d=v-j\text{ or } v-j-1$$

if $3v-t=4j+2$

$$d=v-j-1$$

if $3v-t=4j+3$

Four examples of minimal coverings of type $4\times 4\times 6$

(the digits show the height of elements of $I'$)
Let $\mathcal{G}$ be a covering of $V \times V \times T$, and we put

$$a_{ij} = \# \{(i, j, k) \in \mathcal{G} | 1 \leq k \leq t\}, \quad x_i = \sum_{k=1}^{v} a_{ik}, \quad y_j = \sum_{m=1}^{v} a_{mj}.$$ 

Then we have

if $a_{ij} = 0$, then $x_i + y_j \geq t$.

We shall prove the following lemma.

Lemma. Let $a_{ij}$ be non-negative integer; $1 \leq i, j \leq v$. We assume that $\{a_{ij}\}$ satisfies the following condition * for an integer $t$; $v \leq t \leq 3v$.

if $a_{ij} = 0$, then $x_i + y_j \geq t$. 

Then

$$\sum_{i,j=1}^{v} a_{ij} \geq v^2 - \left[ \frac{(3v-t)^2}{8} \right].$$

Hereafter we shall denote $\sum a_{ij}$ as the abbreviation of $\sum_{i,j=1}^{v} a_{ij}$.

When the equality holds, for the case I which $x_i + y_j \geq t$ for all $i$ and all $j$, we have that
for any \( i \); for the case II which \( x_i + y_j < t \) for some \( i \) and some \( j \), we have that by the suitable replacement of row numbers and column numbers

\[
\begin{align*}
    a_{ij} &= 1 & \text{for } 1 \leq i, j \leq d \\
    a_{ij} &= 0 & \text{for } 1 \leq i \leq d, \; d+1 \leq j \leq d \\
    a_{ij} &= 0 & \text{for } d+1 \leq i \leq v, \; 1 \leq j \leq d \\
\end{align*}
\]

where

\[
\begin{align*}
    d &= v-p & \text{if } 3v-t=4p \\
    d &= v-p & \text{if } 3v-t=4p+1 \\
    d &= v-p \text{ or } v-p-1 & \text{if } 3v-t=4p+2 \\
    d &= v-p-1 & \text{if } 3v-t=4p+3.
\end{align*}
\]

Furthermore

\[
\begin{align*}
    x_i + y_j &= t & \text{for } 1 \leq i \leq d, \; d+1 \leq j \leq d \\
    x_i + y_j &= t & \text{for } d+1 \leq i \leq v, \; 1 \leq j \leq d.
\end{align*}
\]

2. Proof of Lemma

Choose a minimal arrangement \( \{a_{ij}\} \), that is, \( \Sigma a_{ij} \) is minimum in the arrangement satisfying the condition \(*\). Then we have

\[
\Sigma a_{ij} \leq v^2 - \left[ \frac{(3v-t)^2}{8} \right].
\]

We shall denote the transformation of \( \{a_{ij}\} \) by Trans. I which transforms the left arrangement to the right arrangement, as below. Trans. I fix \( \Sigma a_{ij}, \; x_i, \) and \( y_j \) for all \( i \)

\[
\begin{align*}
    \cdots \cdots \cdots & \Rightarrow \cdots \cdots \cdots \\
    u \cdots w & \Rightarrow u+1 \cdots w-1 \\
    \cdots \cdots \cdots & \Rightarrow \cdots \cdots \cdots \\
    v \cdots x & \Rightarrow v-1 \cdots x+1 \\
    \cdots \cdots \cdots & \Rightarrow \cdots \cdots \cdots \\
\end{align*}
\]

Similarly we shall denote the transformation of \( \{a_{ij}\} \) by Trans. II which transforms the left arrangement to the right arrangement, as below. Trans. II fix \( \Sigma a_{ij} \), but does not always fix \( x_i \) and \( y_j \).

\[
\begin{align*}
    \cdots \cdots \cdots & \Rightarrow \cdots \cdots \cdots \\
    u \cdots w & \Rightarrow u+1 \cdots w-1 \\
    \cdots \cdots \cdots & \Rightarrow \cdots \cdots \cdots \\
    \cdots \cdots \cdots & \Rightarrow \cdots \cdots \cdots \\
\end{align*}
\]
Let $A$ be the set of the minimal arrangements $\{a_{ij}\}$. We put

\[ m = \min \{ \# \{(i, j) | a_{ij} = 0\}, \{a_{ij}\} \in A \} \]

\[ M = \{ \{a_{ij}\} \in A | m = \# \{(i, j) | a_{ij} = 0\} \} \]

\[ n = \max \{ \sum_{a_{ij} = 0} (x_i + y_j) | \{a_{ij}\} \in M \} \]

\[ N = \{ \{a_{ij}\} \in M | \sum_{a_{ij} = 0} (x_i + y_j) = n \}. \]

Now from this time we assume that $\{a_{ij}\}$ is an element of $N$.

1) When $a_{ij} = 0$, each component of $i$-th row is equal to 0 or 1, or each component of $j$-th column is equal to 0 or 1.

Proof. Assume that $a_{ik}$ and $a_{mj}$ are larger than one for some $k$ and $m$. Change $a_{ij} + 1$ for $a_{ij}$, $a_{ik} - 1$ for $a_{ik}$, $a_{mj} - 1$ for $a_{mj}$, and $a_{mk} + 1$ for $a_{mk}$, then the new arrangement satisfies the condition of Lemma, and the number of the 0-components in this new arrangement is smaller than that of $\{a_{ij}\}$. This is contrary to the choice of $\{a_{ij}\}$.

2) When $a_{ij} \geq 2$, there are $h$ and $k$ where $a_{ih} = 0$ and $a_{hk} = 0$.

Proof. If all components of $i$-th row and $j$-th column are positive, we may change $a_{ij} - 1$ for $a_{ij}$ with the condition of Lemma. This is contrary to the minimality of $\sum a_{ij}$.

Next assume that all components of $j$-th column are positive, and $a_{ik} = 0$ for some $k$. Change $a_{ij} - 1$ for $a_{ij}$, and change $a_{ik} + 1$ for $a_{ik}$. Then the new arrangement satisfies the condition of Lemma, and the number of 0-components of this arrangement is smaller than that of $\{a_{ij}\}$. This is contrary to the choice of $\{a_{ij}\}$.

3) When $a_{ij} \geq 2$, $a_{ik} = 0$, $a_{hj}$ and $x_h + y_j \geq t$, then $a_{hh} = 0$

Proof. Assume $a_{hh} \geq 1$. Change $a_{ij} - 1$ for $a_{ij}$, $a_{ih} + 1$ for $a_{ih}$, $a_{hj} + 1$ for $a_{hj}$, and $a_{hh} - 1$ for $a_{hh}$, then the new arrangement satisfies the condition of Lemma, and the number of 0-components of this arrangement is smaller than that of $\{a_{ij}\}$. This is contrary to the choice of $\{a_{ij}\}$.

4) If $x_i + y_j < t$, then $a_{ij} = 1$.

Proof. If $a_{ij} \geq 2$, then $a_{ik} = 0$ for some $k$ from (2), and each component of $k$-th column is equal to 0 or 1 from (1). Furthermore $y_k > y_j$, because $x_i + y_k \geq t$ and $x_i + y_j < t$. Therefore $a_{hh} = 1$ and $a_{hj} = 0$ for some $h$. Since $x_h + y_j \geq t$ and $x_i + y_j < t$, $x_h > x_i$. Thus we have $x_h + y_k > x_i + y_k \geq t$. This is contrary to (3).

Hereafter we set $x_1 \leq x_2 \leq \ldots \leq x_v$, and $y_1 \leq y_2 \leq \ldots \leq y_v$ by the suitable replacement of row numbers and column numbers.
(5) If \( a_{i\varphi} \neq 1 \) and \( a_{i\varphi} = 0 \), then \( k < q \).

Proof. Assume \( y_k = y_\varphi \), since all components of \( k \)-th column are 0 or 1, \( a_{u\varphi} = 1 \) and \( a_{u\varphi} = 0 \) for some \( u \). This is contrary to (3), because \( x_u + y_k \geq x_u + y_\varphi \geq t \).

(6) If \( x_i + y_j < t \), then \( a_{ij} = 1 \).

Proof. Assume \( a_{ij} = 1 \) for some \( q \). Then from (3), \( a_{ih} = a_{h\varphi} = 0 \) for some \( k \) and \( h \). We have \( y_k < y_\varphi \) from (5). Since \( x_i + y_j < t \), we have \( y_i < y_k \). Since all components of \( k \)-th column are 0 or 1, \( a_{ih} = 1 \) for some \( k \) larger than \( y_i \). Therefore for some \( u \) smaller than \( u_i \), \( a_{ui} = 0 \). Thus \( t \leq x_u + y_i \leq x_u + y_1 \). Also since \( x_u + y_k > x_u + y_1 \geq t \), \( a_{i\varphi} = 1 \), \( a_{i\varphi} = 0 \), and \( a_{u\varphi} = 1 \), we have \( a_{i\varphi} = 1 \) from (3).

Change \( a_{i\varphi} = 1 \) for \( a_{ij} = 1 \), \( a_{ik} = 1 \) for \( a_{uk} = 1 \), and \( a_{u\varphi} = 1 \) for \( a_{u\varphi} = 1 \). Then the new arrangement satisfies the condition of Lemma. Since \( x_i + y_j < t \), \( a_{ij} = 1 \), \( a_{ik} = 1 \), and \( a_{u\varphi} = 1 \), we have \( a_{ij} = 1 \) for all \( p \).

(7) Proof of Lemma.

Proof of Case I. Assume that \( x_i + y_j \geq t \) for all \( i \) and \( j \).

\[
2v \sum a_{ij} = \sum (x_i + y_j) \geq vt.
\]

Therefore \( \sum a_{ij} \geq vt/2 \). So

\[
v^2 - \left[ \frac{(3v-t)^2}{8} \right] \geq \sum a_{ij} \geq \frac{vt}{2} = v^2 - \frac{(3v-t)^2}{8} + \frac{(t-v)^2}{8}.
\]

When \( t > v \), then \( t = v+1 \) or \( v+2 \), and the equality holds. Since \( \sum a_{ij} = \sum x_i = \sum y_i = vt/2 \), and \( x_i + y_j \geq t \) for all \( i \) and \( j \), we have \( x_i = y_i = t/2 \) for all \( i \).

When \( t = v \)

\[
v^2 - \left[ \frac{v^2}{2} \right] \geq \sum a_{ij} \geq \frac{v^2}{2}.
\]

If \( v \) is even, the equality holds, and we can similarly prove Lemma. When \( v = 2k+1 \),

\[
(2k+1)^2 - \left[ \frac{(2k+1)^2}{2} \right] = 2k^2 + 2k + 1 \geq \sum a_{ij}.
\]

Therefore we may assume that \( x_i \leq k \) for some \( i \). Since \( x_i + y_j \geq 2k+1 \), \( y_j \geq k + 1 \) for all \( j \). Thus \( 2k^2 + 2k + 1 \geq \sum a_{ij} = \sum y_j \geq v(k+1) = 2k^2 + 3k + 1 \). This is a contradiction.

Proof of Case II. Assume that \( x_i + y_j < t \) for some \( i \) and some \( j \),

\[
d = x_1 \leq x_2 \leq \cdots \leq x_v, \quad e = y_1 \leq y_2 \cdots \leq y_v, \quad \text{and} \quad d \leq e.
\]

From (6), we have
Therefore all equalities hold, that is,

\[ \sum_{a_{ik} = l} y_k = d^2, \quad y_k = d \text{ for all } k \text{ where } a_{ik} = 1. \]

Similarly,

\[ x_m = d \text{ for all } m \text{ where } a_{ml} = 1. \]

Since \( x_i + y_k = 2d < t \), we have \( a_{ik} = 1 \) from (4) for all \( m \) and \( k \) where \( a_{ml} = 1 \) and \( a_{lk} = 1. \)

Thus by the suitable replacement of row numbers and column numbers we can obtain the arrangement \( \{a_{ij}\} \) satisfying the assertions of Lemma. For this arrangement, Trans. I and Trans. II can occur in the row numbers and the column numbers larger than \( d \), and cannot occur in others. Thus we can prove that all of the minimal arrangements satisfy the assertions of Lemma.

3. Proof of Theorem

Assume \( \Gamma \) be a minimal covering of \( V \times V \times T \). Then we have

\[ |\Gamma| \leq v^2 - \left[ \frac{(3v-t)^2}{8} \right]. \]

Let \( \{a_{ij}\} \) be the arrangement induced by \( \Gamma \) where

\[ a_{ij} = \#(i, j, k) \in |\Gamma| \mid 1 \leq k \leq t). \]

The arrangement \( \{a_{ij}\} \) satisfies the condition of Lemma, and \( \sum_{a_{ij} = 1} v = |\Gamma| \leq v^2 - \left[ \frac{(3v-t)^2}{8} \right]. \)

Therefore from Lemma, \( |\Gamma| = v^2 - \left[ \frac{(3v-t)^2}{8} \right] \), and \( \{a_{ij}\} \) is a minimal arrangement.

Thus we can conclude that \( \{a_{ij}\} \) satisfies the assertions of Lemma by the suitable replacement of row numbers and column numbers. This arrangement \( \{a_{ij}\} \) is induced by the corresponding replacement of row numbers and column numbers of \( V \times V \times T \).

Now we set the subsets of \( T \) for \( u ; 1 \leq u \leq v \), as follows.

\[ R_u = \{z \mid (u, y, z) \in \Gamma \}, \quad C_u = \{z \mid (x, u, z) \in \Gamma \}. \]

First we shall investigate \( \Gamma \) when \( \{a_{ij}\} \) is case II of Lemma. Since \( a_{ij} = 0 \) and \( x_i + y_j = t \) for \( 1 \leq i \leq d, \quad d + 1 \leq j \leq v \), we have
And $R_i = \{1, 2, \ldots, t\} \setminus C_i$ for any $i$; $1 \leq i \leq d$, and $C_j = \{1, 2, \ldots, t\} \setminus R_j$ for any $j$; $d + 1 \leq j \leq v$. Therefore if $(i, j, k)$ is the element of $\Gamma$ then $1 \leq i$, $j \leq d$ and $k \in R_i$, or $d + 1 \leq i$, $j \leq v$ and $k \in C_v$. Thus we can prove that $\Gamma$ is the union of two 3-directed coverings.

Next we shall investigate $\Gamma$ when $\{a_{ij}\}$ is the case 1 of Lemma.

(1) When $a_{ij} = a_{ki} = a_{ik} = 0$, then $a_{hk} = 0$.
Proof. Since $a_{ij} = 0$ and $x_i + y_j = t$

$R_i \cup C_j = \{1, 2, \ldots, t\}$, $R_i \cap C_j = \phi$.

Similarly

$R_h \cup C_i = \{1, 2, \ldots, t\}$, $R_h \cap C_i = \phi$,

$R_i \cup C_h = \{1, 2, \ldots, t\}$, $R_i \cap C_h = \phi$.

So $R_h = R_i$, $C_h = C_i$. Therefore $R_h \cap C_h = R_i \cap C_i = \phi$. Thus we have that $a_{hk} = 0$.

(2) When $t = v = 2k$, we have the following arrangement by the suitable replacement of row numbers and column numbers from (1) and $x_i = y_i = \frac{t}{2}$ for all $i$.

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There exists the covering corresponding to the above arrangement, and this covering satisfies the conditions of Theorem.

(3) When $t = v + 1$ and $a_{ij} \leq 1$ for all $i$ and $j$, we have the following arrangement by the suitable replacement of row numbers and column numbers.
If \( x = 1 \), then \( t = \frac{(v+1)}{2} = v \), that is, \( v = 1 \) and \( t = 2 \). This is a trivial covering. If \( x = 0 \), then \( t = \frac{(v+1)}{2} = v - 1 \), that is, \( v = 3 \) and \( t = 4 \). For this case, we have the following arrangement:

\[
\begin{array}{ccc}
1 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 1 \\
\end{array}
\]

But we can not construct the covering corresponding to the above arrangement.

(4) When \( t = v+1 \) and \( k_{i,j} \geq 2 \) for some \( i \) and \( j \), we have the following arrangement by the suitable replacement of row numbers and column numbers.

\[
\begin{array}{ccc}
2, 1, & , , , , , , , , , , 1 & 0, , , , , , , , , , 0 \\
1, 2, & , , , , , , , , , , 1 & , , , , , , , , , , \\
, , , , , , , , , , , , , , , , & , , , , , , , , , , \\
1, , , , , , , , , , , , , , , , & 2, 1, , , , , , , , , , \\
1, , , , , , , , , , , , , , , , & 1, 2, , , , , , , , , , \\
0, , , , , , , , , , , , , , , , & 0, , , , , , , , , , \\
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0, , , , , , , , , , , , , , , , & 0, , , , , , , , , , \\
\end{array}
\]
There exists the covering corresponding to the above arrangement, and this covering satisfies the conditions of Theorem.

(5) When \( t = v + 2 \) and \( a_{ij} \leq 1 \) for any \( i \) and \( j \), we have the following arrangement by the suitable replacement of row numbers and column numbers.

\[
\begin{array}{cccc}
1, \ldots, 1 & 1, 1 & 0, \ldots, 0 & 0, \ldots, 0 \\
\ldots & \ldots & \ldots & \ldots \\
1, \ldots, 1 & 1, 1 & 0, \ldots, 0 & \ldots, 0 \\
1, \ldots, 1 & w, x & 1, \ldots, 1 & 1, \ldots, 1 \\
1, \ldots, 1 & y, z & 1, \ldots, 1 & 1, \ldots, 1 \\
0, \ldots, 0 & 1, 1 & 1, \ldots, 1 & 1, \ldots, 1 \\
0, \ldots, 0 & 1, 1 & 1, \ldots, 1 & 1, \ldots, 1 \\
\end{array}
\]

If \( w + x = 2 \), then \( \frac{t}{2} = 1 + \frac{v}{2} = v \), that is, \( v = 2 \) and \( t = 4 \). There exist the trivial coverings corresponding to this arrangement.

If \( w + x = 1 \), then \( \frac{t}{2} = 1 + \frac{v}{2} = v - 1 \), that is, \( v = 4 \) and \( t = 6 \). For this case, we have the following arrangement.

\[
\begin{array}{cccc}
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 \\
\end{array}
\]

We can construct the exceptional covering of Theorem corresponding to the above arrangement.

If \( w + x = 0 \), then \( t = 1 + \frac{v}{2} = v - 2 \), that is, \( v = 6 \) and \( t = 8 \). For this case, we have the following arrangement. But we can not construct the covering corresponding to this arrangement.
(6) When \( t = v + 2 \) and \( a_{ij} \geq 2 \) for some \( i \) and \( j \), we have three types of arrangements by the suitable replacement of row numbers and column numbers from (1).

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If \( x = 1 \), then \( t = \frac{(v + 2)}{2} = v \), that is, \( v = 2 \) and \( t = 4 \). This is impossible. If \( x = 0 \), then \( t = \frac{(v + 2)}{2} = v - 1 \), that is, \( v = 4 \) and \( t = 6 \). For this case, we have the following arrangement.

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<tr>
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<th>2</th>
<th>1</th>
<th>0</th>
<th>0</th>
</tr>
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<tbody>
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<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
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<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>
But we can not construct the covering corresponding to the above arrangement.

\[
\begin{array}{ccc}
\frac{v}{2} & \frac{v}{2} & \frac{v}{2} \\
2, 1, \ldots, 1 & 0, \ldots, 0 & 0, \ldots, 0 \\
1, 2, \ldots, 1 & \ldots, \ldots, \ldots & \ldots, \ldots, \ldots \\
\ldots, \ldots, 1 & \ldots, \ldots, \ldots & \ldots, \ldots, \ldots \\
1, \ldots, 2, 1 & 0, \ldots, 0 & 0, \ldots, 0 \\
1, \ldots, 1, 2 & \ldots, \ldots, \ldots & \ldots, \ldots, \ldots \\
0, \ldots, 0, 0 & \ldots, \ldots, \ldots & \ldots, \ldots, \ldots \\
\end{array}
\]

\(\Gamma\) is the union of two 3-directed covering of type \(\frac{v}{2} \times \frac{v}{2} \times \frac{(v+2)}{2}\).

\[
\begin{array}{ccc}
\frac{(v-2)}{2} & \frac{(v+2)}{2} & \frac{(v-2)}{2} \\
\ast, \ldots, \ldots, \ast & 0, \ldots, 0 & 0, \ldots, 0 \\
\ldots, \ldots, \ldots & \ldots, \ldots, \ldots & \ldots, \ldots, \ldots \\
\ast, \ldots, \ldots, \ast & 0, \ldots, 0 & 0, \ldots, 0 \\
0, \ldots, 0, 0 & \ldots, \ldots, \ldots & \ldots, \ldots, \ldots \\
\end{array}
\]

\(\Gamma\) is the union of 3-directed coverings of type \(\frac{v+2}{2} \times \frac{v+2}{2} \times \frac{v+2}{2}\) and type \(\frac{v-2}{2} \times \frac{v-2}{2} \times \frac{v+2}{2}\), and this covering satisfies the conditions of Theorem.