Rotating, Magnetic, Radiatively Driven Stellar Winds.

I. Basic Equations

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Abstract

A model of a radiatively driven, magnetohydrodynamic wind flow confined in the equatorial plane of the star is formulated and the basic equations are preliminarily analyzed. It is shown that major singular points of the basic equations are slow magnetosonic, the Alfvénic and fast magnetosonic points. The two magnetosonic points are modified by the radiation terms. When the star does not rotate at a slower velocity than at its break-up velocity, the slow magnetosonic point never exists outside the star.

Key words: Hydromagnetics; Rotation; Singular points; Stellar winds

1. Introduction

Weber and Davis (1967) developed a steady-state model of the solar-wind flow including the effects of pressure gradients, gravitation, centrifugal and magnetic forces and showed the spin-down of the Sun over its main-sequence life-time. This model has been applied to studies of various stellar winds: Saito (1974) and Limber (1974) applied the model to Be stars, Belcher and MacGregor (1976) did it to the early Sun, Nerney (1980) estimated upper bounds on magnetic forces in outflows from the stars, Hartmann and MacGregor (1982) calculated protostellar angular momentum loss, and Friend and Mac Gregor (1984) and Nerney and Suess (1987) obtained radiatively driven wind solutions.

Although pioneering work of Weber and Davis played important roles for progress in studies of stellar winds as stated above, it was based on a partial analysis of the basic equations. Three singular points, i.e., slow magnetosonic, the Alfvénic and fast magnetosonic points, were found, these two magnetosonic points were saddle-type singularities and the Alfvénic point was a higher order singularity. The analysis has been enlarged by different authors (e.g., Goldreich and Julian 1970; Yeh 1976; Saito and Saitō...
but not sufficiently been done yet. Nevertheless, Saito and Saito have tried an effective way for a complete analysis of the radial equation of motion and expressed all solutions about the Alfvénic point in explicit form. Next, Weber and Davis obtained super-fast mode solutions which passed through the three singularities. Such solutions were also gotten by the other authors but Limber and Saito. Limber got sub-fast mode solutions which do not go through the fast magnetosonic points but the slow magnetosonic and Alfvénic points and Saito did another sub-fast mode solutions which only go through the Alfvénic points or do not go through all the singularities. These facts indicate that the solution topologies of the basic equations must be clarified on the basis of a complete analysis and different wind solutions including the above solutions must be closely classified thereby.

Nerney and Suess (1987) have recently extended Weber and Davis’s model so as to include forces due to continuum and line radiations in optically thick limit. The extended basic equations can be applied to studies of more different stellar winds than the original equations. In the present paper is taken the first step towards a complete analysis of the extended equations. The equations are written in differential form and integrated for future convenience. Furthermore, an analysis of the modified singular points are preliminarily carried out and when the star is rapidly rotating, the modified slow and fast magnetosonic points are estimated.

2. Basic Equations

In order to describe a steady-state wind flow from the uniformly rotating magnetic star, a body-fixed, spherical coordinate \((r, \theta, \phi)\) whose origin and azimuthal plane are at the center of the star and in its equatorial plane, respectively, is introduced. The fluid is approximated as an inviscid, perfectly conducting polytropic gas.

The magnetohydrodynamic equations are the same as given in Weber and Davis except that radiative bulk forces are added:

\[ \nabla \cdot \mathbf{v} = 0, \]
\[ \rho (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla P + \rho \frac{G M}{r} + \frac{1}{c} \mathbf{J} \times \mathbf{B} + \rho \mathbf{f}_B, \]
\[ P = P_0 \left( \frac{\rho}{\rho_0} \right)^\gamma, \]
\[ \nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J}, \]
\[ \nabla \cdot \mathbf{B} = 0, \]
\[ \nabla \times \mathbf{E} = 0 \]
\[ E + \frac{1}{c} v \times B = 0. \]  

(7)

Here \( \rho \) is the mass density, \( v \) is the wind velocity, \( P \) is the gaseous pressure, \( G \) is the gravitational constant, \( M \) is the stellar mass, \( c \) is the speed of light, \( J \) is the electric current density, \( B \) is the magnetic field strength, \( f_R \) is the radiation force per gram, \( \gamma \) is the polytropic index, \( E \) is the electric field strength and a subscript "0" refers to the equatorial surface of the star. When \( f_R \) is given, the magnetohydrodynamic equations form a closed system. For convenience, \( f_R \) is separated into its continuum and line contributions, i.e., \( f_c \) and \( f_l \). The continuum contribution is due to Thomson scattering of its continuum photons by electrons:

\[ f_c = \frac{s_\text{e} \mathcal{L}}{4\pi c r^2} \hat{e}_r \]

(8)

where \( s_\text{e} \) is the electron scattering opacity per gram, \( \mathcal{L} \) is the stellar luminosity and \( \hat{e}_r \) is unit vector in the radial direction (as seen later, \( \hat{e}_\phi \) and \( \hat{e}_\theta \) are unit vectors in the azimuthal and meridional directions). For line contribution is used an approximate law derived in the original theory of Castor, Abbott, and Klein (1975):

\[ f_l = \frac{s_\text{e} \mathcal{L}}{4\pi c r^2} k t^{-\delta} \hat{e}_r \]

(9)

together with

\[ t = s_\text{e} \rho v_{th} \left| \frac{dv_r}{dv} \right|^{-1} \]

(10)

where \( k \) is a constant which is a measure of the number of strong lines, \( \delta \) is a constant which measures the relative mixing of optically thick and thin lines and \( v_{th} \) is the thermal velocity of the ions which absorb and scatter radiation. For optically thick limit, the value of \( \delta \) is unity.

The kinetic velocity field is assumed to be confined in the equatorial plane and to be under azimuthal symmetry. The line of magnetic force which is believed to start from a magnetic monopole source located at the center of the photosphere and rotating in the equatorial plane, also is under the same symmetry. Then \( v \) and \( B \) are written as

\[ v = v_r(r) \hat{e}_r + v_\phi(r) \hat{e}_\phi \]

(11)

and

\[ B = B_r(r) \hat{e}_r + B_\phi(r) \hat{e}_\phi. \]

(12)

From equation (7) and expressions (11) and (12), it becomes evident that \( E \) only has the meridional component:

\[ E = E \hat{e}_\phi \]

(13)
Therefore electric neutrality arises everywhere because

\[ \nabla \cdot \mathbf{E} = 4\pi \rho_e = 0 \]  

(14)

where \( \rho_e \) is the charge density. The neutrality also does for a relativistic case (Kennel et al. 1983). Since all distance scales of interest are larger than the Debye radius everywhere (Landau and Lifshitz 1958), such neutrality is usually realized.

Suess and Nerney (1973) showed that the meridional velocity \( v_\theta \) and its gradient \( \partial v_\theta / \partial \theta \) can be important quantities under the actions of the Lorentz force and Poynting's vector and the flow is made to diverge away from the equatorial plane. Although a vindication of Weber and Davis's (1967) model is provided by the work of Pizzo et al. (1983), who find an azimuthal motion consist with the Weber and Davis prediction, it is not always insufficient to neglect \( v_\theta \) and \( \partial v_\theta / \partial \theta \). Generally speaking, this model serves as a qualitative guide to the onset and dynamics of significantly magnetic acceleration.

3. Integrations of the Basic Equations

Integrations of equations (1) and (5) are

\[ r^2 \rho v_r = r_0^2 \rho_0 v_{r0} \]  

(15)

and

\[ r^2 B_r = r_0^2 B_{r0}, \]  

(16)

respectively. By using equation (7), equation (6) integrates to

\[ r(v_r B_r - v_\theta B_\theta) = r_0(v_r B_{r0} - v_\theta B_{\theta0}). \]  

(17)

In a frame that rotates with the star, one should substitute \( -\Omega r^2 B_r \) for the above right-hand side where \( \Omega \) is the angular velocity of the roots of the lines of magnetic force. Differences between the two expressions originate in a particular choice of reference level location and do not change the essential properties of the above equation (Balker and Marlborough 1982; MacGregor and Pizzo 1983).

Using equation (4), (8), (9), (10), (15) and (16) and setting

\[ \Gamma = \frac{s_0 \mathcal{L}}{4\pi GM} \]  

(18)

and

\[ \Lambda = \frac{k \mathcal{L}}{4\pi c \rho_0 r_0^2 v_{r0} v_{th}}, \]  

(19)

the radial and azimuthal components of equation (2) reduce to
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\[
(1 - A) v_r \frac{dv_r}{dr} = -\frac{1}{\rho} \frac{dP}{dr} - \frac{GM(1 - \Gamma)}{r^2} + \frac{v_\phi^2}{r} - \frac{1}{8\pi \rho r^2} \frac{d}{dr} (r B_\phi)^2
\]  

and

\[
\frac{d}{dr} (r v_\phi) = \frac{B_{\rho 0}}{4\pi \rho_0 v_\phi} \frac{d}{dr} (r B_\phi),
\]

respectively. Here \( \Gamma \) that is the Eddington parameter does not exceed unity because in the interior of the star, net acceleration must be inward (Cassinelli 1979), but it does usually not attain unity. On the other hand, estimating the values of \( A \) is difficult because \( k \) is model dependent and \( v_{th} \) depends on both the base temperature and the mass of the ions, but \( A < 1 \) has been found by Nerney and Suess (1987) in fast magnetic rotator theories in the presence of strong, nonradiative forces in the acceleration regions. Then including a nonradiative case, \( 0 \leq \Gamma < 1 \) and \( 0 \leq A < 1 \) are set. Thus the equation (20) is mathematically equivalent to that of Weber and Davis. Equation (21) integrates to

\[
r v_\phi - \frac{B_{\rho 0}}{4\pi \rho_0 v_\phi} r B_\phi = L
\]

where \( L \) is a constant which is expressed as the sum of the angular momentum per unit mass and the magnetic torque per unit mass and is thus called as the stream total angular momentum per unit mass. When equations (15), (16) and (17) are substituted into equation (22), radial and azimuthal Alfvénic Mach numbers,

\[
M_{Ar} = \frac{v_r}{v_{Ar}} = \frac{(4\pi \rho)^{1/2} v_r}{B_r},
\]

\[
M_{A\phi} = \frac{v_\phi}{v_{A\phi}} = \frac{(4\pi \rho)^{1/2} v_\phi}{B_\phi},
\]

are introduced together with the radial Alfvén velocity \( v_{Ar} \) and the azimuthal Alfvén velocity \( v_{A\phi} \), the total angular momentum is written as

\[
L = r_0 v_{\phi 0} \left(1 + \frac{1}{M_{Ar 0} M_{A\phi 0}}\right).
\]

Equation (20) is integrated after \( v_\phi \) and \( v_{A\phi} \) are expressed as functions of \( r \) and \( v_r \) by using equations (15), (16), (17), (22) and (25). The expressions are

\[
v_\phi = \frac{L r (v_r - v_{Ar})}{r^2 v_r - r_A^2 v_{Ar}}
\]

and

\[
v_{A\phi} = \frac{L (v_{Ar} v_r)^{1/2} (r^2 - r_A^2)}{r_A (r^2 v_r - r_A^2 v_{Ar})}
\]

at all points but the Alfvénic point and
\[ v_\phi = - \frac{L}{2r_A} \]  

(28)

at the Alfvénic point. Here,

\[ \frac{r_A}{r_0} = \left( \frac{M_{A\phi,0} + M_{A,r,0}}{M_{A\phi,0} + M_{A,r,0}} \right)^{1/2} \quad \text{and} \quad \frac{v_{rA}}{v_{r0}} = \left( \frac{r_0}{M_{A\phi,0} r_A} \right)^2. \]  

(29)

Expression (28) is derived under the conditions of \( M_{A,r} = M_{A\phi} = 1 \) at \( r = r_A \) and \( v_r = v_{rA} \). It is noted that \( r_A/r_0 \leq 1 \) and \( v_{rA}/v_{r0} \leq 1 \) or \( r_A/r_0 > 1 \) and \( v_{rA}/v_{r0} > 1 \) according as \( M_{A\phi,0} \geq 1 \) or \( M_{A\phi,0} < 1 \).

Substituting equations (3), (15), (24), (26) and (27) into equation (20), we obtain

\[ F(r, v_r) \frac{d \ln v_r}{d \ln r} = G(r, v_r), \]  

(30)

\[ F(r, v_r) = \left[ c_{s0} \left( \frac{r_0^2 v_{r0}}{r^2 v_r} \right)^{\gamma - 1} - (1 - \gamma) v_r^2 \right] \left( r^2 v_r - r_A^2 v_{rA} \right)^3 + v_{rA} \left[ \frac{L r v_r}{r_A} (r^2 - r_A^2) \right]^2, \]  

(31)

and

\[ G(r, v_r) = \left[ \frac{G M(1 - \Gamma)}{r} - 2 c_{s0} \left( \frac{r_0^2 v_{r0}}{r^2 v_r} \right)^{\gamma - 1} \right] \left( r^2 v_r - r_A^2 v_{rA} \right)^3 - (Lr)^2 (v_r - v_{rA}) \left( (v_r - v_{rA}) (r^2 v_r - r_A^2 v_{rA}) - 2 v_{rA} v_r (r^2 - r_A^2) \right) \]  

(32)

where \( c_{s0} \) is the speed of sound. Expression (31) is a modified geometric damping factor. Although equation (30) is an ordinary differential equation of first order and \( \gamma \)-dependent degree, it integrates to the following form:

\[ \frac{1}{2} (1 - \gamma) v_r^2 + \frac{1}{2} v_\phi^2 - \frac{G M(1 - \Gamma)}{r} \frac{L}{r_A^2} \frac{r v_{rA} v_\phi}{v_r} - E \]

\[ = c_{s0}^2 \times \begin{cases} \ln \left( \frac{r^2 v_r}{r_0^2 v_{r0}} \right) & \text{for } \gamma = 1, \\ \frac{1}{1 - \gamma} \left( \frac{r_0^2 v_{r0}}{r^2 v_r} \right)^{\gamma - 1} & \text{for } 1 < \gamma \leq 5/3 \end{cases} \]  

(33)

where \( E \) is a constant which is determined under the boundary condition and if \( v_r = v_{r0} \) at \( r = r_0 \), then \( E \) implies the total energy per unit mass. Namely, the first term plus the second, the third and forth terms on the left-hand side of this equation indicate the modified kinetic energy, the effective gravitational energy and the magnetic torque energy, respectively and the right-hand side does the internal potential of the fluid. Since \( \gamma \) generally takes any rational number in its allowed range, the equation is a transcendental equation and \( v_r \) cannot be algebraically determined, but numerically done. The calculation is
difficult for the sake of many-valuedness. The equation is utilized for deciding physical quantities at some special points and infinity (e.g., the gradient of a rectilinear line going through the Alfvénic point and the terminal speeds of wind solutions). In order to determine \( v_r \) at arbitrary points except for the special points, equation (30) must be numerically solved.

4. Brief Descriptions of the Singular Points

Extended Weber and Davis's model gives rise to the singular points in the radial equation of motion. These points are determined by setting expressions (30) and (31) equal to zero,

\[
F(r_q, v_{rq}) = 0 \quad (34)
\]

and

\[
G(r_q, v_{rq}) = 0 \quad (35)
\]

where a subscript "q" refers to the individual singularities.

One of the solutions is obviously the Alfvénic point \((r_h, v_{rh})\). Since this point also is only one singularity of equation (33), it is a higher order singularity. The remaining solutions, if exist, are conjectured from the characteristics of equation (34) in the case of \( \Lambda = 0 \). Then, since the left-hand side of this equation is a geometric factor in the original model, the equation expresses the characteristic condition of disturbances that occur in the magnetofluid (Alfvén and Fälthammer 1963; Limber 1974). Therefore, two solutions, i.e., the slow and fast mode ones, satisfy the above condition. When \( \Lambda \neq 0 \), the two points are changed to a modified slow mode point (denoted by \((r_s, v_{rs})\)) and a modified fast mode point (denoted by \((r_f, v_{rf})\)). It is noted that the two points are no longer singular in equation (33). For reference, when \( \Lambda = \Gamma = 0 \), the other two or three singularities were found by Yeh (1976), but these have no physically important implications.

For future analysis, the modified slow and fast mode points are preliminarily examined in an outflow from the rapidly rotating star of great interest. Eliminating the pressure terms between equations (34) and (35), we obtain

\[
\frac{GM(1-\Gamma)}{r_q} = 2(1-\Lambda)v_{rq}^2 + \left[ u_{\phi}\left(1 + \frac{M_{\phi,0}}{M_{\phi,0}}\right)\frac{r_q}{r_0}\right]^2 \left[ 1 + \frac{(r_h^2 - r_s^2)v_{rq}^2}{r_q^2 v_{rq} - r_h^2 v_{rh}^2} \right]. \quad (36)
\]

Since \( r_s < r_h \), an inequality

\[
\frac{GM(1-\Gamma)}{r_s} > \left[ u_{\phi}\left(1 + \frac{M_{\phi,0}}{M_{\phi,0}}\right)\frac{r_s}{r_0}\right]^2 \quad (37)
\]

holds. This inequality is rewritten as
where \( v_{esc0} = \left(2GM/r_0\right)^{1/2} \) is the escape velocity. If \( \Gamma = 0 \) and \( v_\phi = v_{esc0} \) (named the Keplerian circular velocity or the break-up velocity), \( r_s/r_0 < 1 \) is always gotten. According as \( \Gamma \) increase, the inequality holds even if \( v_\phi \) is somewhat slower than the break-up velocity. Next, if \( v_\phi \) considerably decreases, \( r_s/r_0 > 1 \) arises, still its value is close to that of \( r_{s,max}/r_0 \).

Then, since \( r_s^4 \ll r_A^4 \) and \( r_s^2 v_r \ll r_A^2 v_A \), equation (36) leads to

\[
u_r \approx \frac{v_\phi}{v_M} \left(1 + \frac{M_A r_0}{M_A r_0} \right) \frac{r_{s,max}}{r_0} \left[ 1 - \frac{r_A}{r_{s,max}} \right]^{1/2}
\]

This expression is a new relation between \( r_s \) and \( v_r \). When the value of \( 1 - r_s/r_{s,max} \) is appropriately given as an initial guess, the value of \( v_r \) is immediately calculated.

In contrast to the slow mode point, the fast mode point is usually thought to be far beyond the Alfvénic point for rapid rotation (e.g., Nerney and Suess 1987), namely: \( r_f > r_A \). Then equation (36) leads to

\[
u_f \approx \frac{v_M}{(1 - A)^{1/3}} \left[ 1 + \frac{1 - \Gamma}{12(1 - A)^{1/3}} \left( \frac{v_{esc0}}{v_M} \right)^2 \frac{r_0}{r_f} \right]^{1/3}
\]

where \( v_M \) is the Michel velocity (1969) in the fixed coordinate:

\[v_M = \left[ v_r v_\phi \left( \frac{1}{M_A r_0} + \frac{1}{M_A r_0} \right) \right]^{1/3} \]

The second term on the right-hand side of equation (40) is different from the corresponding term in Nerney and Suess, who found the term of the order of \( r_f^{-2} \).

On the basis of the above analysis, the detailed analysis of the major singularities will be made in a subsequent paper. The bahaviors of various solutions will be also clarified.

References