Supplement to the Paper "Commutative Semigroups Obtained by an Abelian Group and a Generalized $\mathcal{F}$-Function"

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This is a correction and supplement to the previous paper [3]. In [3] we showed that $\mathcal{F}$-semigroups can be classified into the three types and gave three examples to them. But the third example $S=(G:/'}$ in pp. 3-4 was in the wrong, because the verification on $\mathcal{F}$-function $I'$ was not complete. In the present note we shall correct to the third example and shall determine the structure of the corrected new example.

1. Correction to the third examle.

p. 3, 4th line from the bottom
"abelian group in which contains an infinite cyclic subgroup" is replaced by "infinite cyclic group generated by an element $\alpha$"

p. 3, 1st and 2nd lines from the bottom

$I'(\alpha, \beta) = \begin{cases} 1 & \text{if at least one of } \alpha \text{ and } \beta \text{ equals } \varepsilon, \\ 0 & \text{otherwise.} \end{cases}$

is replaced by

$I'(\alpha^i, \alpha^j) = \begin{cases} 1 & \text{if } ij = 0, \\ 0 & \text{if } ij > 0, \\ \min \{ |i|, |j| \} - 1 & \text{if } ij < 0 \text{ and } i + j = 0, \\ \min \{ |i|, |j| \} & \text{if } ij < 0 \text{ and } i + j \neq 0, \end{cases}$

where $|i|$ means the absolute value of an integer $i$.

To make sure we shall restate the corrected new example below.

Let $G$ be an infinite cyclic group generated by an element $\alpha$. And consider a function $I' : G \times G \rightarrow \mathbb{Z}$ such that

$I'(\alpha^i, \alpha^j) = \begin{cases} 1 & \text{if } ij = 0, \\ 0 & \text{if } ij > 0, \\ \min \{ |i|, |j| \} - 1 & \text{if } ij < 0 \text{ and } i + j = 0, \\ \min \{ |i|, |j| \} & \text{if } ij < 0 \text{ and } i + j \neq 0, \end{cases}$

where $|i|$ means the absolute value of an integer $i$.

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2. Verification on the new example.

It is obvious that \( I' (\alpha^i, \alpha^j) = I' (\alpha^i, \alpha^j) \in \mathbb{Z}_+ \) for all \( \alpha^i, \alpha^j \in G \) and that \( I' (\alpha^i, \alpha^j) \neq 0 \) and there exists an \( \alpha^i \in G \) such that \( \sum_{j=1}^{s} I' (\alpha^i, (\alpha^i)^j) < I' (\alpha^i, \alpha^j) \) for all \( s \in \mathbb{Z}_+ \), since \( \sum_{j=1}^{s} I' (\alpha^i, \alpha^j) = 0 \) for all \( s \in \mathbb{Z}_+ \).

We now show that \( I' (\alpha^i, \alpha^j) + I' (\alpha^i + \alpha^k) = I' (\alpha^i, \alpha^j + \alpha^k) + I' (\alpha^i, \alpha^k) \) for all \( \alpha^i, \alpha^j, \alpha^k \in G \). Assume that \( i \leq j \leq k \) without loss of generality. The following cases are considered.

1. \( i = j = k \),
2. \( i j k = 0 \),
3. \( 0 < i \leq j \leq k \),
4. \( 0 < i < j < k \),
5. \( i \leq j < 0 < k \),
6. \( i \leq j \leq k < 0 \).

We prove only the case (4). The case (5) is similarly shown and the remaining cases are easy. In the case (4), there are the following subcases:

1. \( |i| \leq j \),
2. \( j < |i| < k \),
3. \( k \leq |i| \).

Case (i). It follows that \( I' (\alpha^i, \alpha^j) + I' (\alpha^i + \alpha^k) = I' (\alpha^i, \alpha^j + \alpha^k) + I' (\alpha^i, \alpha^k) = |i| \).

Case (ii). Since \( k - |i + j| = k - |i| + j > 0 \), it follows that \( I' (\alpha^i, \alpha^j) + I' (\alpha^i + \alpha^k) = j + |i + j| = |i| \) and \( I' (\alpha^i, \alpha^j + \alpha^k) + I' (\alpha^i, \alpha^k) = |i| \).

Case (iii). It follows that \( k - |i + j| = j + k - |i| > 0 \) and

\[
I' (\alpha^i, \alpha^j) + I' (\alpha^i + \alpha^k) = \begin{cases} |i| & \text{if } k - |i + j| > 0, \\ j + k & \text{if } k - |i + j| = 0, \\ |i| & \text{if } j + k - |i| > 0, \\ j + k & \text{if } j + k - |i| < 0. \end{cases}
\]

Hence we get \( I' (\alpha^i, \alpha^j) + I' (\alpha^i + \alpha^k) = I' (\alpha^i, \alpha^j + \alpha^k) + I' (\alpha^i, \alpha^k) \) for all \( \alpha^i, \alpha^j, \alpha^k \in G \). Therefore \( I' \) is an \( \mathcal{F} \)-function on \( G \) and \( S = \langle G : I' \rangle \) is an \( \mathcal{F} \)-semigroup of Type 2.

3. The structure of the new \( \langle G : I' \rangle \).

Let \( p_0 \) be the smallest semilattice congruence relation on the foregoing new \( S = \langle G : I' \rangle \) and let \( S_0, S_1 \) and \( S_2 \) be the \( p_0 \)-classes containing \( (\alpha^0, 0) \), \( (\alpha, 0) \) and \( (\alpha^{-1}, 0) \) respectively. Then these \( p_0 \)-classes are given by \( S_0 = \{ (\alpha^0, 0) \} \cup \{ (\alpha, s) : \alpha \in G, s \in \mathbb{Z}_+ \} \), \( S_1 = \{ (\alpha^i, 0) : i \in \mathbb{Z}_+ \} \) and \( S_2 = \{ (\alpha^{-i}, 0) : i \in \mathbb{Z}_+ \} \). Hence \( S = S_0 \cup S_1 \cup S_2 \), and so \( S \) is the union of the semilattice \( A = \{ 0, 1, 2 \} \) of \( \mathcal{F} \)-semigroups \( S_1, \lambda \in A \).

It is easily seen that \( A \) is a lower semilattice such that \( 0 \leq 1, 0 \leq 2, 1 \leq 2 \) and \( 2 \leq 1 \) under the ordering \( \leq \) defined by \( \mu \leq \lambda \) if and only if \( S_1 S_0 \subseteq S_0 \) and that both \( S_1 \) and \( S_2 \) are isomorphic onto the additive semigroup \( \mathbb{Z}_+ \) of all positive integers. For any \( \alpha^k \in G \) we define \( p(\alpha^k) \) by
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We now put $I_0 (\alpha^i, \alpha^j) = I (\alpha^i, \alpha^j) - p (\alpha^i, \alpha^j)$ for all $\alpha^i, \alpha^j \in G$, where $p (\alpha^i, \alpha^j) = p (\alpha^i + \alpha^j) - p (\alpha^i) - p (\alpha^j)$. Then it follows that

$$I_0 (\alpha^i, \alpha^j) = \begin{cases} 1 & \text{if } ij \geq 0, \\ \min \{ |i|, |j| \} + 1 & \text{if } ij < 0, \end{cases}$$

so $I_0 (\alpha^i, \alpha^j) \in \mathbb{Z}$ and that $I_0 (\alpha^i, \alpha^j) + I_0 (\alpha^i + \alpha^j, \alpha^k) = I_0 (\alpha^i, \alpha^j + \alpha^k)$ for all $\alpha^i, \alpha^j, \alpha^k \in G$. In fact it holds that

$$I_0 (\alpha^i, \alpha^j) + I_0 (\alpha^i + \alpha^j, \alpha^k) = I' (\alpha^i, \alpha^j) + I' (\alpha^i + \alpha^j, \alpha^k) - p (\alpha^i + \alpha^j) + p (\alpha^i) + p (\alpha^j)$$

$$+ p (\alpha^i + \alpha^j) = I' (\alpha^i, \alpha^j + \alpha^k) + I' (\alpha^i, \alpha^k) - p (\alpha^i + \alpha^j + \alpha^k) + p (\alpha^i) + p (\alpha^j + \alpha^k)$$

$$I_0 (\alpha^i, \alpha^j + \alpha^k) + I_0 (\alpha^i, \alpha^j).$$

Hence $I_0$ is an $\mathcal{F}$-function on $G$. Let $(G; I_0)$ be the $\mathcal{R}$-semigroup constructed from $G$ and $I_0$. Then the $\rho_0$-class $S_0$ is isomorphic onto $(G; I_0)$. To show this we take $(\alpha^0, 0) \in S_0$. Then the set of all prime elements with respect to $(\alpha^0, 0)$ is $((\alpha^i, p (\alpha^i)) : \alpha^i \in G)$ and it holds that

$$(\alpha^i, p (\alpha^i)) (\alpha^j, p (\alpha^j)) = (\alpha^i + \alpha^j, p (\alpha^i + \alpha^j)) (\alpha^0, 0) I_0 (\alpha^i, \alpha^j).$$

Hence the structure group $G^*$ of $S_0$ and the $\mathcal{F}$-function $I^*$ on $G^*$ with respect to $(\alpha^0, 0)$ are given by $G^* = ((\alpha^i, p (\alpha^i)) : \alpha^i \in G)$ and $I^* ((\alpha^i, p (\alpha^i)) \rho, (\alpha^j, p (\alpha^j)) \rho) = I_0 (\alpha^i, \alpha^j)$. The above $(\alpha^i, p (\alpha^i)) \rho$ denotes the congruent class containing $(\alpha^i, p (\alpha^i))$, where $\rho$ is a congruence relation on $S_0$ defined by $\xi \rho \eta$ if and only if $\xi = (\alpha^0, 0)^s \eta$ or $\eta = (\alpha^0, 0)^s \xi$ for some $s \in \mathbb{Z}$. Since $G^*$ is isomorphic onto $G$ under the mapping $(\alpha^i, p (\alpha^i)) \rho \mapsto \alpha^i$, it follows that $S_0 \cong (G^*; I^*) \cong (G; I_0)$. Thus we have the following result.

$S = \{ G; I' \}$ is the semilattice union of $\mathcal{R}$-semigroups $S_\lambda$, $\lambda \in \Lambda$ satisfying the following conditions:

1. $\Lambda = \{ 0, 1, 2 \}$ is a lower semilattice such that $0 \leq 1$, $0 \leq 2$, $1 \leq 2$ and $2 \leq 1$.
2. $S_0 \cong (G; I_0)$ and $S_1 \cong S_2 \cong \mathbb{Z}_+$. Conversely, assume that a semilattice $\Lambda = \{ 0, 1, 2 \}$ and the set $\{ S_\lambda : \lambda \in \Lambda \}$ of $\mathcal{R}$-semigroups $S_\lambda$ corresponding each element $\lambda \in \Lambda$ are given and that the above conditions (1) and (2) are satisfied. Then the union $S$ of $S_\lambda$, $\lambda \in \Lambda$ is a semilattice union and is isomorphic onto the $\mathcal{R}$-semigroup $\{ G; I' \}$ which was given as the new example.

To prove this, we put $S_0 = (G; I_0)$, $S_1 = \mathbb{Z}_+$ and $S_2 = \mathbb{Z}_-$ (the additive semigroup of all negative integers) and $S = S_0 \cup S_1 \cup S_2$ and define a binary operation $(\ast)$ on $S$ as follows: For $m, n \in S_1$, $(-m), (-n) \in S_2$, $(\alpha^i, s), (\alpha^j, t) \in S_0$,

$$m \ast n = m + n,$$

$$(-m) \ast (-n) = (-m) + (-n),$$
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\[ m \cdot (-n) = (\alpha^{m-n}, I_0 (\alpha^m, \alpha^{-n}) - 2), \]
\[ m \cdot (\alpha^i, s) = (\alpha^{m+i}, I_0 (\alpha^m, \alpha^i) + s - 1), \]
\[ (-m) \cdot (\alpha^i, s) = (\alpha^{-m+i}, I_0 (\alpha^{-m}, \alpha^i) + s - 1), \]
\[ (\alpha^i, s) \cdot (\alpha^i, t) = (\alpha^i, s) (\alpha^i, t). \]

It is obvious that \( S \) is closed under the operation (\( \cdot \)) and that (\( \cdot \)) is commutative.

For any \( m \in S_1 \), \( (-n) \in S_2 \) and \( (\alpha^i, s) \in S_0 \) we have
\[ (m \cdot (-n)) \cdot (\alpha^i, s) = (\alpha^{m-n+i}, I_0 (\alpha^{m-n}, \alpha^i) + I_0 (\alpha^m, \alpha^{-n}) + s - 2) \]
\[ = (\alpha^{m-n+i}, I_0 (\alpha^m, \alpha^{-m+i}) + I_0 (\alpha^m, \alpha^i) + s - 2) \]
\[ = m \cdot ((-n) \cdot (\alpha^i, s)). \]

The remaining cases are similarly shown, and hence (\( \cdot \)) is associative.

Setting \( I (\alpha^i, \alpha^i) = I_0 (\alpha^i, \alpha^i) + p (\alpha^i, \alpha^i) \) for all \( \alpha^i, \alpha^i \in G \), it follows that
\( I \) is equal to the \( \mathcal{F} \)-function \( I' \) on \( G \). Define a mapping \( \phi \) of \( S \) to the \( \mathcal{F} \)-semigroup \( \langle G : I' \rangle \) as follows:
\[ \phi : \xi \mapsto \begin{cases} 
(\alpha^m, 0) & \text{if } \xi = m \in S_1, \\
(\alpha^{-m}, 0) & \text{if } \xi = (-m) \in S_2, \\
(\alpha^i, p(\alpha^i) + s) & \text{if } \xi = (\alpha^i, s) \in S_0.
\end{cases} \]

Then \( S \) is isomorphic onto the \( \mathcal{F} \)-semigroup \( \langle G : I' \rangle \) under the mapping \( \phi \). In fact
\[ (m \cdot (\alpha^i, s)) \phi = (\alpha^{m+i}, p(\alpha^{m+i}) + I_0 (\alpha^m, \alpha^i) + s - 1) \]
\[ = (\alpha^{m+i}, p(\alpha^i) + I' (\alpha^m, \alpha^i) + s) \]
\[ = m \phi (\alpha^i, s) \phi. \]

The remaining cases are also similarly shown, so that \( \phi \) is an isomorphism. Thus the proof has been finished.

References