On primitive rank 5 permutation groups with two doubly transitive suborbits of same sizes

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P.J. Cameron [2], T. Ito [3] and M. Numata [4] have shown that if $G$ is a primitive rank 5 permutation group on a finite set $\Omega$, and the stabilizer $G_a$ of a point $a$ of $\Omega$ is doubly transitive on $\Gamma_1(\alpha)$ and $\Gamma_2(\alpha)$ where $\Gamma_1(\alpha)$ and $\Gamma_2(\alpha)$ are two $G_a$-orbits, and $\Gamma_1 \circ \Gamma_1 \neq \Gamma_2 \circ \Gamma_2$, then $G$ is isomorphic to the small Janko simple group and $|\Omega|=266$.

For the case of $\Gamma_1 \circ \Gamma_1 = \Gamma_2 \circ \Gamma_2$, we have two examples. (see [2])

The purpose of this paper is to prove the following theorem.

Theorem. Let $G$ be a primitive rank 5 permutation group on a finite set $\Omega$. Let $\Gamma_0, \Gamma_1, \Gamma_2, A$ and $\Sigma$ be $G$-orbits in $\Omega \times \Omega$. ($\Gamma_0$ is trivial $G$-orbit) Suppose that the stabilizer $G_a$ of a point $a$ of $\Omega$ is doubly transitive on $\Gamma_1(a)$ and $\Gamma_2(a)$, and not doubly transitive on $A(a)$ and $J(a)$, and $\Gamma_1 \circ \Gamma_1 = \Gamma_2 \circ \Gamma_2 = A$ and $|\Gamma_1(a)| = |\Gamma_2(a)|$.

Then we have

1) $\pi_1 = \pi_2$ ($\pi_i$; permutation character of $G_a$ on $\Gamma_i(\alpha)$, $i=1,2$)
2) $\Gamma_1$ and $\Gamma_2$ are paired to each other
3) $\Gamma_1 \circ \Gamma_1 = \Gamma_2 \cup \Sigma$

(see [4] for notations)

Proof. We put $v = |\Gamma_1(\alpha)| = |\Gamma_2(\alpha)|$, $|A(\alpha)| = v(v-1)/k$ and $|\Sigma(\alpha)| = s$.

If $\pi_1 \neq \pi_2$, then $\Gamma_1 \circ \Gamma_2$ is a $G$-orbit in $\Omega \times \Omega$, and $|\Gamma_1 \circ \Gamma_2(\alpha)| = v^2m$, $m>1$ because $\Gamma_1 \circ \Gamma_1 = \Gamma_2 \circ \Gamma_2$. Count in two ways quadrilaterals $(\alpha, \beta_1, \gamma, \beta_2)$ whose edges are successively $\Gamma_1, \Gamma_2, \Gamma_1^*$ and $\Gamma_1$, and $\beta_1 \neq \beta_2$.

Then we have

$$|\Omega| = \frac{v^2}{m} m(m-1) = |\Omega| \frac{v(v-1)}{k} k.$$

So

$$v(m-1) = (v-1)k.$$

Since $k \leq (v-1)/2$, this is impossible. Thus we have $\pi_1 = \pi_2$ and $\Gamma_1 \circ \Gamma_2$ is the union of two $G$-orbits in $\Omega \times \Omega$. If $\Gamma_1 \circ \Gamma_2 \neq \Delta \cup \Sigma$, then $\Gamma_1 \circ \Gamma_2 \supset \Gamma_1$, or $\Gamma_2$. Therefore $\Gamma_1$ and $\Gamma_2$ are paired to each other. Assume $\Gamma_1 \circ \Gamma_2 = \Gamma_1 \circ \Gamma_1 = \Gamma_2 \cup \Delta$, we have that $G$ is a

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rank 4 permutation group by P. J. Cameron [1, Prop. ]. This is a contradiction. Thus we have 2) and 3).

Now we assume $\Gamma_1 \cup \Gamma_2 = \Delta \cup \Sigma$. Let $I$, $C_1$, $C_2$, $D$ and $S$ be the basis matrices of $\Gamma_1, \Gamma_2, \Delta$ and $\Sigma$, respectively. We put
\[ C_1^*C_2^* = xD + yS. \]
(for $\alpha, \delta \in \Delta$, $x = \#\{(\alpha, \gamma) \in \Gamma_1, (\delta, \gamma) \in \Gamma_2\}$.
for $\alpha, \xi \in \Sigma$, $y = \#\{(\beta, \beta) \in \Gamma_1, (\xi, \beta) \in \Gamma_2\}$)

Then we have
\[ v^2 = v(v-I)x/k + yS. \]  \hspace{1cm} (1)

Now
\[ (C_1^*C_1)C_2^* = (vI+kD)C_2^* = vC_2^* + k(v-I)C_2^* + \text{terms not involving } C_2^* \]
\[ C_1^*(C_1C_2)^* = C_1^*(xD + yS) = [(v-I)x/k + yS/v]C_2^* + \text{terms not involving } C_2^*. \]

So we have
\[ (v-I)x^2/k + sy^2/v = v+k(v-I). \]  \hspace{1cm} (2)

From (1)
\[ s = \frac{v^2k - v(v-I)x}{ky} \]  \hspace{1cm} (3)

(2) and (3) yield
\[ y = \frac{(v-I)(k^2-x^2)+kv}{vk-vx+x}. \]  \hspace{1cm} (4)

So
\[ k(v-y) = (v-I)(k-x)(y-k-x). \]  \hspace{1cm} (5)

We put for $\alpha, \gamma \in \Delta$
\[ |\Delta(\alpha) \cap \Gamma_1(\gamma) | = z_1, \quad |\Delta(\alpha) \cap \Gamma_2(\gamma) | = z_2. \]

Let $A_{r_1}$ be the intersection matrix of $\Gamma_1$ for basis matrices $I$, $C_1$, $C_2$, $D$ and $S$.

\[ A_{r_1} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
v & 0 & 0 & k & 0 \\
0 & 0 & 0 & x & y \\
0 & 0 & \frac{(v-I)x}{k} & z & \frac{(v-k-x-z_1)d}{s} \\
0 & v-1 & sy/v & v-k-x-z_1 & v-y \frac{(v-k-x-z_1)d}{s}
\end{bmatrix}. \]
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Since $A_{r_1}A_{r_2}^* = A_{r_2}A_{r_1}^*$. count the entries $(2,4)$ and $(3,4)$, we have

$$z_2^k = z_1x + (v - k - x - y)z_1$$
$$z_1^k = z_2x + (v - k - x - y)z_2$$

So

$$(z_2 - z_1)k = (z_2 - z_1)(y - x).$$

If $z_2 \neq z_1$, $k + x - y = 0$. From (5) $y = v$. This is impossible. Since $z_2 = z_1$, we have

$$z_1 = z = \frac{y(v - k - x)}{k + x + y}.$$  \hspace{1cm} (6)

From (3), (4) and (6), we have

$$TrA = z + v - y - \frac{(v - k - x - y)z}{s} = v - \frac{2ky}{y + k - x}.$$  

We shall show that $y + \frac{(v - k - x - y)z}{s} - z = \frac{2ky}{y + k - x}$.

Now

$$y + \frac{(v - k - x - y)(v - k - x)ky}{k(v - k - x + y)} - \frac{y(v - k - x)}{k + x + y} - \frac{2ky}{y + k - x}$$

Multiply the above formula by $(vk - vx + x) (k - x + y) / y$,

$$(vk - vx + x)(k - x + y) + (v - 1)(k - x)(v - k - x) - (v + k - x)(vk - vx + x)$$
$$= (vk - vx + x)(k - x + y - v - k - x) + (vk - vx + x - h)(v - k - x)$$
$$= (vk - vx + x)(-v + y + v - k - x) - k(v - k - x)$$
$$= (vk - vx + x)(y - k - x) - k(v - k - x)$$
$$= (v - 1)(k - x)(y - k - x) + k(y - k - x) - k(v - k - x)$$
$$= k(-y) + k(-v + y) = 0$$ (by (5))

Thus we have $TrA = v - \frac{2ky}{y + k - x} = v - 2k - \frac{2kx - 2k^2}{y + k - x}$. Therefore $\frac{2kx - 2k^2}{y + k - x}$ is an integer.
\[
\frac{2kx - 2k^2}{y + k - x} = \frac{2k(x - k)(vk - vx + x)}{(v - 1)(k^2 - x^2) + kv + vk^2 - vxx + kx - vxx + vx^2 - x^2} \quad \text{(by (4))}
\]

\[
= \frac{2(x - k)(vk - vx + x)}{(2v - 1)(k - x) + v}
\]

\[
= (x - k) + \frac{(k - x)(v - x - k)}{(k - x)(2v - 1) + v}
\]

From \(v/2 > k > x\), we have

\[0 < (k - x)(v - x - k) < (k - x)(2v - 1) + v.\]

So \(\frac{(k - x)(v - x - k)}{(k - x)(2v - 1) + v}\) is not an integer. This is impossible. Thus we conclude that the case of \(\Gamma_1 \circ \Gamma_2^* = A \cup \Sigma\) does not occur.

References

[1] Peter J. Cameron: Primitive groups with most suborbits doubly transitive, Geometriae Dedicata 1 (1973), 434-446.

