Green's operators on multi-almost product manifolds

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Introduction. In the present paper, we shall study about Green's operators on a multi-almost product manifold. A multi-almost product structure is one of generalizations of an almost product structure defined in [1,3], and it seems to be closely related to a multifoliate structure defined by K. Kodaira and D. C. Spencer [2].

B. L. Reinhart [3] studied harmonic integrals on an almost product manifold. In [3], operators $d'$, $\delta'$ and $\Delta'$ are given with respect to $x$ alone and a Green's operator $G'$ is defined and an analogue to Hodge's theorem is proved, i.e., it is proved that the $d'$ cohomology is isomorphic to the space of $\Delta'$ harmonic forms. But the Green's operator $G'$ is not completely continuous and the kernel of $\Delta'$ is infinite dimensional.

In § 1, we shall state the definition of multi-almost product structure, which is defined by giving $m$ fields of subspaces of tangent spaces of the given manifold. And operators $d_s$ $(s=1,2,\ldots,m)$ and $\Delta_s = d_s \delta_s + \delta_s d_s$ ($\delta_s$ is the adjoint operator of $d_s$).

In § 2, we shall construct Green's operators for $\Delta_K$.

§ 1 Multi-almost product structure

Let $M$ be a connected paracompact (real) $n$-dimensional differentiable manifold of class $C^\omega$. Denote its tangent bundle by $T(M)$, the tangent $P$-vectors by $A^pT(M)$ $(p=1,2,\ldots,n)$ or $T^p(M)$, and a bundle of Grassmann algebras by $A^pT(M) = \sum_{p=0}^n T^p(M)$, where $A^pT(M)$ is trivial bundle of real numbers, and $A^pT(M) = T(M)$. Let $\Phi(M)$ be the bundle of cotangents, with other notations, $A^p\Phi(M)$, $A\Phi(M)$, etc., analogous to those for $T(M)$. Let $d$ denote the operation of exterior differentiation on $A\Phi(M)$.

A multi-almost product structure is defined on $M$ by giving $m$ fields $S_1, S_2, \ldots, S_m$ (we assume that the integer $m$ is in $0 < m \leq n$), of class $C^\omega$, of complementary proper subspaces of $T_x$. If we set $n_s = \dim S^s$ $(s=1, 2, \ldots, m)$, then we have $n = \sum_s n_s$, where we assume $n_s \neq 0$ for all $s$. Let $P^s$ $(s=1, 2, \ldots, m)$ be the projection of $T_x$ onto $S^s_x$ at every point $x \in M$, where $S^s_x$ denotes the value of $S^s$ at $x$.

In particular, if $m = 2$, the multi-almost product structure on $M$ reduces to an almost product structure in the sense of [1,3]. A manifold with multi-almost product structure is said to be a multi-almost product manifold. Let $M$ be a multi-almost product manifold. A multi-almost product structure on $M$ induces the projection operators
\[ H_{s_1, s_2, \ldots, s_m} (\text{resp. } \Pi_{s_1, s_2, \ldots, s_m}) \text{ in } \Lambda T(M) \text{ (resp. } A \Phi) \]. A vector field which is a cross-section of \( H_{s_1, s_2, \ldots, s_m} \) is said to be of type \((s_1, s_2, \ldots, s_m)\) and similarly for \( A \Phi(M) \). A map \( f \) of \( M \) into itself is said to be of type \((t_1, t_2, \ldots, t_m)\) with respect to the structure if

\[ H^*_{s_1, s_2, \ldots, s_m} \circ f = f^* \circ H_{s_1, s_2, \ldots, s_m} \]

where \( f^* : \Lambda T(M) \to \Lambda T(M) \) is the map induced by \( f \); an operator \( L \) on \( \Lambda T(M) \) or \( A \Phi(M) \) may be decomposed into various types according to the definition:

\[ H_{s_1, s_2, \ldots, s_m} L = \sum_{s_1, s_2, \ldots, s_m} H_{s_1, s_2, \ldots, s_m} \circ L \circ H_{s_1, s_2, \ldots, s_m} \].

Since the operator \( d \) of exterior differentiation is decomposed, so we find that

\[ d = \sum_{s_1, s_2, \ldots, s_m} d_{s_1, s_2, \ldots, s_m}, \]

where \( d_{s_1, s_2, \ldots, s_m} \) is of type \((0, \ldots, 0, -1, 0, \ldots, 2, 0, \ldots, 0)\) for \( \alpha < \beta \), of type \((0, \ldots, 0, 2, 0, \ldots, 0, -1, 0, \ldots, 0)\) for \( \alpha > \beta \), and of type \((0, \ldots, 0, 1, 0, \ldots, 0)\) for \( \alpha = \beta \) respectively.

According to Guggenheim and Spencer [1], we may define an operator \( \Phi^p(M) \to \Phi^{p+1}(M) \) by the axioms:

1. If \( \phi \in A \Phi x_v(M) \), \( v \in T_x(M) \) \( (x \in M) \), then

\[ (d_\phi v)(\phi) = \left( P_\phi v \right)(\phi). \]

2. If \( \phi \in A \Phi x_v(M) \), then

\[ (d_\phi d + dd_\phi) \phi = 0. \]

3. If \( \phi \in A \Phi x_v(M) \) and \( \psi \in A \Phi x_v(M) \), then

\[ (d_\phi \phi \wedge \psi) = d_\phi \phi \wedge \psi + (-1)^p \phi \wedge d_\psi \phi. \]

It is easily verified that \( 2 \sum_{s_1, s_2, \ldots, s_m} d_{s_1, s_2, \ldots, s_m} - \sum_{s_1, s_2, \ldots, s_m} d_{s_1, s_2, \ldots, s_m} \), for fixed \( \alpha \), satisfies these axioms hence we may set

\[ d_\alpha = 2 \sum_{s_1, s_2, \ldots, s_m} d_{s_1, s_2, \ldots, s_m} - \sum_{s_1, s_2, \ldots, s_m} d_{s_1, s_2, \ldots, s_m} \text{ for fixed } \alpha, \]

and find that \( d = \sum_{s_1, s_2, \ldots, s_m} d_{s_1, s_2, \ldots, s_m} \).

In general, the operator \( d_\alpha \) is not differential, i.e., \( d_\alpha^2 \neq 0 \). It is, however, differential if and only if the multi-almost product structure is integrable, by which we mean that the Poisson bracket of two vector fields of type \( P_\alpha \) is again a vector field of the same type. In this case, the Poincaré lemma is true for \( d_\alpha \) and forms of type \((s_1, s_2, \ldots, s_n)\) with \( s_n > 0 \).

Let \( I_\alpha \) be the increasingly ordered \( n_\alpha \)-tuple \((i_1, i_2, \ldots, i_{n_\alpha}) \) \( (\alpha = 1, 2, \ldots, m) \). If a multi-almost product structure on \( M \) is integrable, we can apply the Frobenius' theorem to obtain local coodinates \((x_{i_1}^{(\alpha)}, \ldots, x_{i_{n_\alpha}}^{(\alpha)}, \ldots, x_{i_{n_{m}}^{(\alpha)})})\) in the neighbourhood of any point such that \( P_\alpha T(M) \) is spanned by \((\partial / \partial x_{i_1}^{(\alpha)}) \). This local coordinates is said to be a local multi-
**almost product coordinate system.** In terms of such a coordinate system, a differential form \( \phi \) of type \((s_1, s_2, \ldots, s_a, \ldots, s_m)\) may be written as

\[
\phi = \phi_{I_1(t_1)} dx_{i_1}^{1(t_1)} \wedge \cdots \wedge dx_{i_a}^{a(t_a)} \wedge \cdots \wedge dx_{i_m}^{m(t_m)},
\]

where \(I_a(\alpha)\) is the increasingly ordered \(s_a\)-tuple \((i_1, i_2, \ldots, i_s)\) of integers in \(I_a\) and

\[
dx_{s_a}^{a(t_a)} = dx_{i_1}^{1(t_1)} \wedge \cdots \wedge dx_{i_s}^{s(t_s)}.
\]

We notice that an integrable multi-almost product structure is a special case of a (real) multifoliate structure given by Kodaira and Spencer [2]. In fact, if a multifoliate structure on \(M\) has a finite totally ordered set instead of a finite partially ordered set \(P\) which defines a \(P\)-multifoliate set of integers and then if a local coordinate system \((x_1^{t_1}, \ldots, x_a^{s_a}, \ldots, x_m^{s_m})\) has a property \(\partial x_i^{t_j}/\partial x_i^{t_k} = 0\) for \(\alpha \neq \beta\). The multifoliate structure on \(M\) in this case becomes an integrable multi-almost product structure.

We may introduce on a multi-almost product manifold \(M\) a Riemannian metric such that the subspaces \(P_a T(M)\) \((\alpha = 1, 2, \ldots, m)\) are mutually orthogonal. If a multi-almost product structure on \(M\) is integrable and if local multi-almost product coordinates are used, the metric will have the form

\[
ds^2 = \sum_{i,j=1}^{m} g_{ij}(s) \, dx_i^t \, dx_j^t + \sum_{i,j=1}^{m} g_{ij}^{(s)} \, dx_i^s \, dx_j^s + \cdots + \sum_{i,j=1}^{m} g_{ij}^{(m)} \, dx_i^m \, dx_j^m,
\]

where components of metric tensor \(g_{ij}(s)\) \((\alpha = 1, 2, \ldots, m)\) are \(C^\infty\) functions of \((x_1^{t_1}, x_2^{t_2}, \ldots, x_m^{t_m})\). This metric is said to be a **multi-almost product metric**.

Hereafter we assume that the manifold \(M\) has an integrable multi-almost product structure and a multi-almost product metric.

We may define an operator \(\delta_R\) for a form \(\phi\) with totally degree \(q\) by

\[
\delta_R \phi = (-1)^{nq+n+1} * d * \phi,
\]

where \(\ast\) is the duality operator defined by the given metric. Since \(\delta_R = (-1)^{nq+n+1} * d * \phi\), we have \(\delta = \sum_{a=1}^{m} \delta_a\). Let \(K\) be the increasingly ordered \(p\) tuple \((\lambda, \mu, \nu)\) \((0 < p \leq m; \lambda, \mu, \nu \in \{1, 2, \ldots, m\})\) and by \(K\) we denote the increasingly ordered \((m-p)\)-tuple consisting of the subset of \(\{1, 2, \ldots, m\}\) complementary to \(K\). We may define special Laplacian operators by

\[
\Delta_a = d_a \delta_a + \delta_a d_a \quad (a = 1, 2, \ldots, m),
\]

\[
\tilde{\Delta}_R = \Delta_1 + \Delta_2 + \cdots + \Delta_v,
\]

\[
\tilde{\Delta} = \Delta_1 + \Delta_2 + \cdots + \Delta_m.
\]

It is easily showed that \(\Delta_a, \tilde{\Delta}_R\) and \(\tilde{\Delta}\) preserve types, i.e., they are of type \((0, 0, \ldots, 0)\). We denote a usual Laplacian operator by \(\Delta\).
If the manifold $M$ is compact, the inner product $(\phi, \psi)$ is defined in usual way for $\phi, \psi \in \Phi(M)$ and the norm $\phi = \sqrt{(\phi, \phi)}$ is defined.

Let $M$ be a compact multi-almost product manifold with integrable structure. A Riemannian metric on $M$ is said to be torsionless if $g^{(a)}_{ij}$ $(\alpha = 1, 2, \ldots, m)$ are $C^\infty$ functions of $x^a\alpha$ $(\alpha = 1, 2, \ldots, m)$ alone. Setting
$$d_K = d_\lambda + d_\mu + \cdots + d_s, \quad \delta_K = \delta_\lambda + \delta_\mu + \cdots + \delta_s$$
and
$$\Delta_K = d_K \delta_K + \delta_K d_K,$$
we have the following proposition.

**Proposition 1.** If the metric on $M$ is torsionless, then $\tilde{\Delta}_K = \Delta_K$ holds good and moreover $A = \tilde{A}$.

Since the metric is torsionless we can prove by the method similar to Reinhart [3].

**Proposition 2.** If $M$ is an integrable multi-almost product manifold, we have $d^2_K = 0$ and $d^2_K = 0$.

In fact, for all $\lambda, \mu \in \{1, 2, \ldots, m\}$. On the other hand, $d_a^* = 0$ for all $a \in \{1, 2, \ldots, m\}$ since the structure is integrable. Then the proposition is proved.

Then we have
$$\Delta = A = \Delta_K + \Delta_K$$
for a torsionless integrable multi-almost product manifold $M$.

§ 2 Green's operators for $\Delta_K$

Let $\mathcal{V}_K^r$ be a sheaf of germs of square integrable forms of degree $r$ which depend on $x_K$ alone, where $x_K = \{x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(s)}\}$, and $\mathcal{V}_K^r$ is defined similarly. Let $\mathcal{V}_{K\bar{K}}$ be the direct sum $\sum_{r,s} \mathcal{V}_K^r \wedge \mathcal{V}_{\bar{K}}^s$. By the method similar to Reinhart [3], we can construct a coherent subsheaf $\mathcal{A}_K$ of $\mathcal{V}_{K\bar{K}}$ such that $\phi_1, \phi_2, \ldots, \phi_t$ are sections of $\mathcal{A}_K$, where $\phi_1, \phi_2, \ldots, \phi_t$ are also sections of $\mathcal{V}_{K\bar{K}}$. And any $\phi$ is a stalk of $\mathcal{A}_K$ is expressed as $\sum \phi_i \wedge \gamma_i$ in some neighbourhood, where $\gamma_i$ is one of finite set of sections of $\mathcal{V}_K^r$.

Next we shall define the second inner product of forms on $M$. It is defined by
$$((\phi, \psi)_D = (\phi, \psi) + \sum_{a, j} \int_{V} \sum_{i} \frac{\partial \phi_i(s)}{\partial x_a} \frac{\partial \psi_i(s)}{\partial x_a} dx_a \ dx_K dx_{\bar{K}}$$
and the corresponding norm by $\| \phi \|_D = \sqrt{((\phi, \phi)_D)}$, where $(\phi, \phi)$ is the usual inner
product and if a finite covering \(\{U_i\}\) of \(M\).

The space \(L_0(\mathcal{B}_K)\) is the completion in the usual norm of the set of \(C^\infty\) sections of \(\mathcal{B}_K\); the space \(P_2(\mathcal{B}_K)\) is the completion in the second norm. Let \(H_\alpha(\mathcal{B}_K)\) be a subspace consisting of forms \(\omega\) in \(P_2(\mathcal{B}_K)\) satisfying \(d_\alpha \omega = \bar{d}_\alpha \omega = 0\) and \(H_\alpha(\mathcal{B}_K) = H_\alpha(\mathcal{B}_K) \cap H_{\alpha-1}(\mathcal{B}_K) \cap \cdots \cap H_{\alpha-\alpha}(\mathcal{B}_K)\). We have the following theorems by the method similar to [3] and the proof is omitted.

**Theorem 1.** Let \(M\) be a compact multi-almost product manifold with torsionless metric and let \(\mathcal{B}_K\) be a coherent subsheaf of \(\mathcal{L}_K\). Let \(\omega_0\) be a form in \(L_0(\mathcal{B}_K)\) orthogonal to \(H_{\alpha}(\mathcal{B}_K)\) and \(H_{\alpha}(\mathcal{B}_K) \cap H_{\alpha}(\mathcal{B}_K) \cap \cdots \cap H_{\alpha}(\mathcal{B}_K)\). We have the following theorems by the method similar to [3] and the proof is omitted.

\[
\sum_{\alpha \in \mathbb{K}} \left\{ (d_\alpha \omega_0, d_\alpha \zeta) + (\bar{d}_\alpha \omega_0, \bar{d}_\alpha \zeta) \right\} = (\omega_0, \zeta)
\]

for all \(\zeta \in P_2(\mathcal{B}_K)\). Furthermore, the form \(\omega_0\) to \(\Omega_0\) is a bounded linear transformation from \(L_0(\mathcal{B}_K)\) into \(P_2(\mathcal{B}_K)\).

**Theorem 2.** Let \(M\) be a compact multi-almost product manifold with torsionless metric, and \(\mathcal{B}_K\) be a coherent subsheaf of \(\mathcal{L}_K\). If \((\Omega_0, D_\alpha) = (\omega_0, \zeta)\) for all \(\zeta\) with sufficiently small support in \(L_0(\mathcal{B}_K)\), which are \(C^\infty\), and if \(\omega_0\) is in \(L_0(\mathcal{B}_K)\) and \(C^\infty\), then \(\Omega_0\) is \(C^\infty\).

Combining Theorem 1 and 2, we can construct the Green’s operator for \(D_\alpha\). By the method of M. H. Stone we can show that \(D_\alpha = D_\alpha + \bar{D}_\alpha = D_\alpha\) is a self-adjoint operator, in particular is also closed, hence so is \(D_\alpha\).

**Theorem 3.** There is defined on \(L_0(\mathcal{B}_K)\) a bounded symmetric operator \(G_K, \mathcal{B}_K\) such that \(D_\alpha G_K = G_K D_\alpha\) and \(D_\alpha G_K, \mathcal{B}_K = D_\alpha G_K, \mathcal{B}_K = 0\), where \(H_K(\mathcal{B}_K)\) is the projection to the space \(H_K(\mathcal{B}_K)\).

**Proof.** Let \(\omega_0\) be orthogonal to \(H_K(\mathcal{B}_K) = H_\alpha(\mathcal{B}_K) \cap H_{\alpha-1}(\mathcal{B}_K) \cap \cdots \cap H_{\alpha-\alpha}(\mathcal{B}_K)\). By Theorem 1, we can find \(\Omega_0\) orthogonal to \(H_K(\mathcal{B}_K)\) such that \(\sum (d_\alpha \Omega_0, d_\alpha \zeta) + \sum (\bar{d}_\alpha \Omega_0, \bar{d}_\alpha \zeta) = (\omega_0, \zeta)\) for all \(\zeta \in P_2(\mathcal{B}_K)\). Moreover if \(\omega_0\) is \(C^\infty\) by Theorem 2, \(\Omega_0\) is \(C^\infty\), whence \(D_\alpha \Omega_0 = \omega_0\). Define \(G_K, \mathcal{B}_K(\omega_0) = \Omega_0\), then \(D_\alpha G_K, \mathcal{B}_K \omega_0 = \omega_0\). Let \(H_K(\mathcal{B}_K)\) be a projection on \(H_K(\mathcal{B}_K)\). If \(H_K(\mathcal{B}_K)\) is \(C^\infty\) for any \(\phi\), so that \(H_K(\mathcal{B}_K)\) is \(C^\infty\) if \(\phi\) is. Let \(\phi\) be an arbitrary form in \(L_0(\mathcal{B}_K)\), and let \(\{\phi_i\}\) be a sequence of \(C^\infty\) forms approximating \(\phi\). Then \(\{\omega_i\} = \{\phi_i - H_K, \mathcal{B}_K \phi_i\}\) is a sequence of \(C^\infty\) forms approximating \(\omega = \phi - H_K, \mathcal{B}_K \phi\) and each \(\omega_i\) satisfies \(D_\alpha G_K, \mathcal{B}_K \omega_i = \omega_i\). Since \(\|G_K, \mathcal{B}_K \omega_i\| \leq \|G_K, \mathcal{B}_K \omega_i\| \leq C|\omega_i|\), \(G_K, \mathcal{B}_K\) is bounded. Hence \(G_K, \mathcal{B}_K \omega_i = G_K, \mathcal{B}_K \omega_i\), while \(D_\alpha G_K, \mathcal{B}_K \omega_i = \omega_i\). Since \(D_\alpha\) is closed, we have \(D_\alpha G_K, \mathcal{B}_K \omega = \omega\). Now let us define \(G_K, \mathcal{B}_K \phi = G_K, \mathcal{B}_K \omega\). Then \(D_\alpha G_K, \mathcal{B}_K \phi = \phi - H_K, \mathcal{B}_K \phi\). If \(H_K(\mathcal{B}_K)\), we have \(G_K, \mathcal{B}_K \phi = 0\), hence \(G_K, \mathcal{B}_K G_K, \mathcal{B}_K \phi = 0\). Also \(H_K, \mathcal{B}_K G_K, \mathcal{B}_K \phi = 0\). Hence \(G_K, \mathcal{B}_K\) is symmetric on \(L_0(\mathcal{B}_K)\) and \(G_K, \mathcal{B}_K D_\alpha \phi = -H_K, \mathcal{B}_K \phi\) for all \(\phi\) in the domain of \(D_\alpha\).

By a method similar to Reinhart, we can show that the Green’s operator is independent from the choice of coherent subsheaves of \(\mathcal{L}_K\).

**Theorem 4.** Let \(M\) be a compact multi-almost product manifold with torsionless
metric. Let $\phi$ be a $C^q$ section of the sheaf $\mathfrak{L}_K$. Then there is a $C^q$ section $G_K\phi$ of $\mathfrak{L}_K$ such that $\Delta_K G_K\phi = G_K \Delta_K \phi = -H_K\phi$ and $G_K H_K\phi = H_K G_K\phi$. In these equations, $G_K\phi = G_K, \mathfrak{A}_K\phi$ and $H_K\phi = H_K, \mathfrak{B}_K\phi$, where $\mathfrak{B}_K$ is an arbitrary coherent subsheaf of $\mathfrak{L}_K$ having $\phi$ as a section.

The operator $G_K$ appeared in Theorem 4 is called a Green's operator for $\Delta_K$. $G_K$ is not in general a bounded operator on the Hilbert space of all square integrable forms on $M$, since $G_K$ may be densely defined without being everywhere defined. We can easily show that $\Delta_K d_K = d_K \Delta_K$ and $G_K$ commutes with $d_K$ and $\delta_K$. Hence we have an analogue to Hodge's theorem.

**Theorem 5.** Let $M$ be a compact multi-almost product manifold with torsionless metric. Then the $d_K$ cohomology of sections of the sheaf $\mathfrak{L}_K$ is isomorphic to the space of $\Delta_K$ harmonic sections of $\mathfrak{L}_K$, the isomorphism being given by assigning to each cohomology class the unique harmonic form contained in it.

Next we shall consider a special case. We consider a Green's operator for $\tilde{\Delta}$. In this case, we can construct the Green's operator by a method similar to Reinhart [3], and we can prove the following result.

**Proposition 3.** Let $M$ be a compact multi-almost product manifold. There exists a symmetric and completely continuous operator $\tilde{G}$ which satisfies

$$\beta = \tilde{\Delta} \tilde{G} \beta + \tilde{H} \beta = \tilde{G} \tilde{\Delta} \beta + \tilde{H} \beta$$

and $\tilde{G} H = H \tilde{G} = 0$, where $\tilde{H} \beta$ is the projection of $\beta$ on the space $\tilde{H} = \{\phi \tilde{\Delta} \phi = 0\}$ and $\beta$ is a $C^q$ form in $\mathfrak{L}_2(M)$.

The Green's operator $\tilde{G}$ for $\tilde{\Delta}$ fails to commute with $d, d^*$ and related operators.

**Bibliography**


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