ON $\beta^*$-SEMIGROUPS

Morio Sasaki

(Received August 31, 1969)

Recently T. Tamura [6] has determined all the types of $\beta$-semigroups, where by a $\beta$-semigroup we mean a semigroup $S$ satisfying the following two conditions:

1. Any subset of $S$ which contains a definite element $e$ is a subsemigroup of $S$.
2. Any subsemigroup of $S$ contains $e$.

A semigroup which satisfies (1) is called a $\beta^*$-semigroup. $\beta$-semigroups, Rédei's semigroups [3,4], and left [right] zero semigroups are all $\beta^*$-semigroups.

The author, in [4], has announced the theorem on $\beta^*$-semigroups without proof and he and the other one, in [5], have established the theorem on semigroups satisfying the condition (2). The present paper we want to give the detailed description for [4].

Throughout this paper we denote by $S$ a $\beta^*$-semigroup and by $e$ a definite element of $S$. It is easily shown that any subset of $S$ which contains $e$ is a $\beta^*$-semigroup and that a homomorphic image of $S$ is also a $\beta^*$-semigroup.

1. Equivalence relations on $S$.

It will be clear that $ab=e$ or $a$ or $b$ for every $a, b$ of $S$, since the subset $\{e, a, b\}$ forms a subsemigroup of $S$.

Put $T=\{x \in S; x^2=x\}$, $U=\{x \in S; x^2=e, x \neq e\}$, $ex=xe=e$ and $V=\{x \in S; x^2=e, x \neq e, ex=xe=x\}$. Then we have

Lemma 1. $S=T+U+V$ (disjoint class-sum) and the cardinal number $|V|$ of $V$ is $\leq 1$.

Proof. Let $x \in S \setminus T$, then it follows that $x \in U+V$ since $x^2=ex=xe=e$ or $x$, hence $S \subseteq T+U+V$, and hence $S=T+U+V$. To show $|V| \leq 1$, let $a, b \in V$ and suppose $ab=a$. Then it follows that $b=eb=(aa)b=a(ab)=aa=e$, contradicting $b \in V$, hence $ab \neq a$. Similarly $ab \neq b$. Hence $ab=e$, so we have $a=ae=a(ab)=(aa)b=eb=b$. Thus we have $|V| \leq 1$.

Now we shall define relations $\gamma$ [\$\gamma\$] and $\sim$ as follows:

$a \sim b$ [\$a \gamma b\$] means $ab=a$ and $ba=b$ [\$ab=b$ and $ba=a\$] for $a, b \in T$,

$a \sim b$ does $ab=ba=e$ for $a, b \in S \setminus T$.

Lemma 2. $\gamma$ [\$\gamma\$] and $\sim$ are equivalence relations on $T$ and $S \setminus T$ respectively.

Proof. The reflexivity and symmetry are trivial, so we prove the transitivity only. Let $a \gamma b$ and $b \gamma c$. Then we get $ac=(ab)c=ab=a$ and $ca=(cb)a=c(ba)=cb=c$,
hence \( a \sim c \). Similarly we get \( a \sim b \) and \( b \sim c \) imply \( a \sim c \). And let \( a \sim b \) and \( b \sim c \). If \( a=b \) or \( b=c \), then \( a \sim c \) is clear, so hereafter we suppose that \( a \neq b \) and \( b \neq c \). Then we get \( a \in U \), because if \( a \in U \), then \( a=ea=(ba)a-b(aa)=be=e \) or \( b \), contradicting. Similarly we get \( c \in U \). And since

\[
a = \begin{cases} ac = ac^2 = ae = e & \text{if } ac = a, \\ ca = c^2 a = ea = e & \text{if } ca = a, \\ c = \begin{cases} ac = a^2 c = ec = e & \text{if } ac = c, \\ ca = ca^2 = ec = e & \text{if } ca = c, \end{cases} \end{cases}
\]

we get \( ac = ca = e \), hence \( a \sim c \).

**Lemma 3.** If \( a \neq b \), then any two of \( a \sim b \), \( a \sim b \) and \( a \sim b \) are all incompatible.

*Proof.* It is obvious.

**Lemma 4.** If \( a \neq b \) and \( b \neq c \), then both \( a \sim b \), \( b \sim c \) and \( a \sim b \), \( b \sim c \) are incompatible.

*Proof.* Letting \( a \sim b \) and \( b \sim c \), it follows that

\[
b = ba = b(ac) = (ba)c = bc = c \quad \text{if } ac = a,
\]

\[
b = bc = b(ca) = (bc)a = ca = a \quad \text{if } ca = a.
\]

\[
b = cb = (ac)b = a(cb) = ab = a \quad \text{if } ac = c,
\]

\[
b = ca = c(ab) = ca = c \quad \text{if } ca = c.
\]

Hence we get \( ac = ca = e \), so that \( a = ab = a(cb) = (ac)b = eb = e \) or \( b \), hence \( a = e \). Hence \( b = ba = be = b(ca) = (bc)a = ca = a = e = a \), contradicting. Therefore \( a \sim b \) and \( b \sim c \) are incompatible. Similarly we get \( a \sim b \) and \( b \sim c \) are incompatible.

Define a new relation \( \approx \) as follows:

\( a \approx b \) if and only if at least one of \( a \sim b \), \( a \sim b \) and \( a \sim b \) holds. Then, by Lemma 2, 3 and 4, we have

**Lemma 5.** \( \approx \) is an equivalence relation on \( S \).

Furthermore

**Lemma 6.** For any \( a, b \in U \), any \( c \in T \) and \( w \in V \), it follows that (i) \( a \approx b \), (ii) \( w \neq a \), (iii) \( w \neq c \) and (iv) \( a \neq c \).

*Proof.* (i) Since

\[
a = \begin{cases} ab = ab^2 = ae = e & \text{if } ab = a, \\ ba = b^2 a = ea = e & \text{if } ba = a, \\ \end{cases}
\]

\[
ab = a^2 b = eb = e \quad \text{if } ab = b,
\]

\[
ba = ba^2 = be = e \quad \text{if } ba = b,
\]

we obtain \( ab = ba = e \). Hence \( a \sim b \), and hence \( a = b \).
(ii) It is clear that both \( w \triangleright a \) and \( w \triangleleft a \) do not occur. If \( w \sim a \), then \( w = ew = (aw)w = a(ww) = ae = e \) or \( a \), contradicting. Hence \( w \neq a \).

(iii) and (iv) will be clear.

**Lemma 7.** If \( V \neq \emptyset \), then \( e \sim a \) implies \( e = a \).

**Proof.** Let \( e \sim a \). Then the two cases \( e \triangleright a \) and \( e \triangleleft a \) are possible since \( e \in T \). If \( e \triangleright a \), then for \( w \in V \)

\[
w = ew = (ea)w = e(aw) = \begin{cases} ea = e & \text{if } aw = a, \\
e = e & \text{if } aw = e
\end{cases}
\]

and these contradict with \( w \neq e \). Hence \( aw = w \), and hence \( e = w^2 = (aw)w = aw^2 = ae = a \).

Similarly we obtain that \( e \sim a \) implies \( e = a \).

Thus, we have seen that the equivalence relation \( \approx \) gives the following partition of \( S \):

**Theorem 1.** \( S \) can be represented as

\[
S = \sum_{n \in I} S_n = \sum_{i \in J_1} S_{i} + \sum_{\mu \in J_2} S_{\mu} + \sum_{\nu \in J_3} S_{\nu} \quad \text{(disjoint class-sum)},
\]

where \( \Lambda = J_1 \cup J_2 \cup J_3 \), \( J_3 = \{ w, a, u \} \), \( S_u \), \( \mu \in J_2 \) is a maximal left [right] zero subsemigroup which does not contain \( e \), \( S_u = U \), \( S_u = \emptyset \) or \( \{ w \} \) and \( S_i \) is a maximal left or right zero subsemigroup which contains \( e \), especially \( S_i = \{ e \} \) when \( S_i \neq \emptyset \).

2. **Orderings on \( S \).**

We shall define orderings \( \geq, \succ \) \([\succeq, \succcurlyeq] \) as follows:

\[
a \geq b \text{ means either } a = b \text{ or } ab = ba = a,
\]

\[
a \succ b \quad [a \succeq b] \text{ does } ab = a \text{ and } ba = e \] \( \text{[}ab = e \text{ and } ba = a\] \text{ for } e \neq a, e \neq b, a \neq b.
\]

\( a \succ b \) denotes that \( a \geq b \) and \( a \neq b \).

**Lemma 8.** \( \geq \) is a partial ordering on \( S \).

**Proof.** The reflexive and anti-symmetric laws are trivial. We prove the transitive law. Letting \( a \geq b \) and \( b \geq c \), it holds \( ac = (ab)c = a(bc) = ab = a \) and \( ca = c(ba) = (cb)a = ba = a \), hence \( a \geq c \).

It will be clear that for any \( a \in S \) one and only one of \( e \succ a \), \( e = a \) and \( e \prec a \) holds and that for any \( u \in U \) it follows that \( w \succ e \succ u \).

**Lemma 9.** If \( a \succ b \) \([a \succeq b] \), then

\[
\begin{align*}
e \succ b \text{ and } e \succ a \text{ [}e \succ a\] & \quad \text{if } a \in T, \\
e \succ a, e \succ b \text{ and } a \in U, b \in T & \quad \text{if } a \notin T.
\end{align*}
\]

**Proof.** We shall prove the case where \( a \succ b \) only. Letting \( a \succ b \), it follows that \( eb = (ba)b = b(ab) = ba = e \) and
hence \( e > b \). Now, if \( a \in T \), then \( e = (ba)a = b(aa) = ba = e \) and \( e = a(ba) = (ab)a = aa = e \), hence \( e \sim a \). And if \( a \not\in T \), then \( e = (ba)a = b(aa) = be = e \) and \( e = a(ba) = (ab)a = aa = e \), hence \( e > a \) and hence \( b \in T \), because if not so, it follows \( a = ab - ab - ae = e \), contradicting.

**Lemma 10.** \( a \in E \). \( a \subseteq b \); \( a \subseteq b \); \( a \subseteq b \), \( a \subseteq b \); are all incompatible.

**Proof.** If \( a \subseteq b \) and \( a \subseteq b \), then \( e = (ba)a = b(aa) \), contradicting. And if \( a \subseteq b \) \( \left[ a \subseteq b \right] \) and \( a \subseteq b \), then \( e = ba = a \), contradicting too.

Let us, here, define a new ordering \( \preceq \) as follows:

\( a \preceq b \) if and only if at least one of \( a \subseteq b \), \( a \subseteq b \) and \( a \subseteq b \) holds.

**Lemma 11.** (i) \( a \preceq b \) and \( b \preceq c \) imply \( a \preceq c \) or \( a > c \).

(ii) \( a \preceq b \) and \( b \preceq c \) imply \( a \preceq c \) or \( a > c \).

(iii) \( a > b \) and \( b \preceq c \) imply \( a > c \).

**Proof.** (i) First we shall consider the case where \( a \preceq b \) and \( b \preceq c \left[ b \preceq c \right] \). By Lemma 9, it follows that \( e > b \), \( e > c \) and \( a \in T \), because if \( a \not\in T \), then it holds \( b \in T \), hence \( e \sim b \left[ e \sim b \right] \) and hence \( be = b = e \left[ eb = b = e \right] \), contradicting. Hence \( e \sim a \), and hence \( ac = (ae)c = a(ec) = ae = a \). While, if \( ca = c \), then \( c = ca = c(ea) = (ca)e = ce = c \), contradicting, so we get \( ca = c \) or \( a > c \). Hence \( a > c \) or \( a \preceq c \). Next, let \( a \preceq b \) and \( b \preceq c \). Then it follows that \( ac = (ab)c = a(bc) = ab = a \left[ ac = (ab)c = a(bc) = ae = e \right. \) and \( ca = c(ba) = (cb)a = ea = e \) or \( a[ca = c(ba) = (cb)a = ba = a] \). Hence \( a > c \) or \( a \preceq c \left[ a \preceq c \right. \). But \( a \preceq c \left[ a \preceq c \right. does not occur. In fact, supposing \( a \preceq c \left[ a \preceq c \right. \), we get

\[
\begin{align*}
\text{if } a \not\in T, b \not\in T, \\
\begin{align*}
e &= e = (ba)a = b(ba) = ba = a \\
\left[ e &= ae = a(bb) = (ab)b = ab - a \right. \\
e &= aa = (ba)a = b(aa) = be = b \\
\left[ e &= aa = a(ab) = (aa)b = eb = b \right. \\
e &\sim a \sim b \left[ e \sim a \sim b \right. \text{ if } a \not\in T, b \not\in T, \\
a &\sim b \text{ if } a \not\in T, b \not\in T.
\end{align*}
\end{align*}
\]

and these are all contradictions. Thus the proof has been finished.

**Lemma 12.** \( \preceq \) is a partial ordering on \( S \),
Proof. The reflexivity is clear and by Lemma 11 the transitivity is also so. Now, suppose that \( a \geq b \) and \( b \geq a \). Then we arrive at \( a \geq b \) and \( b \geq a \), hence, by Lemma 8, \( a = b \).

**Lemma 13.** For \( w \in S \) and any \( u \in S \), it holds that
\[
w \geq x \geq e \geq y \geq u \text{ implies } x = e \text{ and } y = e.
\]

**Proof.** Suppose that \( w \geq x \geq e \). If \( w \geq x \geq e \), then \( e > w \) by Lemma 9, contradicting. Hence \( w \geq x \), and hence \( e = w^2 - w^3 = x^2 - y^3 \), that is, \( e \leq x \). While since \( x \geq e \), we get \( x \geq e \). Hence, by Lemma 8, \( a = b \).

**Lemma 14.** Let \( a \in B \). Then it holds that
(i) \( a > b \) implies \( a > e \)
(ii) if there exists \( c \leq e \) such that \( a < c \) and \( b < c \), then it holds \( c = e \).

**Proof.** (i) Since \( a \neq b \), we may assume that \( a \neq T \) without loss of generality. Then we get \( a e = a (ab) = (a b) a = a b = e \). And \( e i = (b i) = b i = i a = e \), hence \( e > x \). Therefore if \( a \neq T \) and \( b \neq T \), then \( e > a \) and \( e > b \) and if \( a \neq T \) and \( b \neq T \), then, since
\[
e = \frac{b a = (b e) a}{e} = ba = b a = b e = b \quad \text{ if } be = b
\]
\[
ab = a (eb) = (ea) b = eb = b \quad \text{ if } eb = b
\]
and these are contradictions, we get \( eb = be = e \), and hence \( e > a \) and \( e > b \).

(ii) If there exists \( c \leq e \) such that \( a < c \) and \( b < c \), then \( c = b c = b (ac) = (ba) c = e c = e \).

**Lemma 15.** Let \( b \neq e \) such that \( a < c \) and \( b < c \), then \( c \neq b c = b (ac) = (ba) c = e c = e \).

**Proof.** (i) It follows that \( ac = (ab) c = a (bc) = ab \). If \( ca = e \), then \( a = ba = (bc) a = b \) \( (ca) = be = b \), contradicting to \( a \neq b \), hence \( ca = e \) or \( a \). By the way, if \( ca = e \), then \( a = ba = (bc) a = b (ca) = be = e \) or \( b \), hence \( a = e \), that is, \( ca = e \) or \( a \). It will be clear that \( a \neq c \). Hence \( a > c \).

(ii) It holds that \( ac = (ab) c = a (bc) = ab = a \) and \( ca = (cb) a = c (ba) = e c = e \) or \( c \). By the way, if \( ca = e \), then \( b = b c = b (ca) = (bc) a = ba = e \), contradicting. Hence \( ca = e \). Here, if \( c = e \), then we arrive at \( b \neq a \), hence \( b = ba = e \), contradicting, and if \( a = c \), then it follows \( b \neq a \), hence \( b = ba = e \), contradicting too. Thus we get \( a \neq c \).
(iii) Since it follows that

\[
\begin{align*}
  a &= \begin{cases} 
    ba = (bc)a = b(ca) = e & \text{if } ca = e \text{ or } c, \\
    ac = a(cb) = (ac)b = ab = e & \text{if } ac = a, \\
    b = bc = (bc)a = (ba)c = ac & \text{if } ac = c,
  \end{cases}
\end{align*}
\]

we arrive at \( ca = a \) and \( ac = e \). Here, if \( a = c \), then \( a^2 = a = e \), contradicting, and if \( c = e \), then \( e \geq b \), while \( e > b \). This is a contradiction. Hence \( a \geq c \).

(iv) If \( a = c \), then, since \( a \neq b \) and \( b \succ c \)(\( = a \)), it holds \( e = ba - ab = b = a \), contradicting. Hence \( a \neq c \). And since

\[
\begin{align*}
  a &= \begin{cases} 
    ca = (cb)a = c(ba) = ce = e & \text{if } ca = a, \\
    ac = a(cb) = (ac)b = ab = e & \text{if } ac = a, \\
    b = bc = (bc)a = (ba)c = ac & \text{if } ac = c,
  \end{cases}
\end{align*}
\]

we arrive at \( ca = ac = e \). If \( c = e \), then, since \( b \prec c \)(\( = e \)) and \( e > b \) by Lemma 14, we get \( b = e \), contradicting. And \( c \in T \) is clear. Thus we get \( a \neq c \).

Similarly we have the following lemma:

**Lemma 16.** Let \( b \succ c \), \( b \neq c \) and \( a \neq b \). Then

(i) \( a > b \) implies \( a > c \),
(ii) \( a \geq b \) implies \( a \geq c \),
(iii) \( a \geq b \) implies \( a \geq c \),
(iv) \( a \neq b \) implies \( a \neq c \).

**Lemma 17.** Let \( b \prec c \), \( b \neq c \) and \( a \neq b \). Then

(i) \( a > b \) implies \( a > c \),
(ii) \( a \geq b \) implies \( a \geq c \),
(iii) \( a \geq b \) implies \( a \geq c \),
(iv) \( a \neq b \) implies \( a \neq c \).

**Proof.** (i) Since \( b \prec c \), it follows that

\[
\begin{align*}
  &ab = ab^2 = ae = a(cb) = (ac)b = eb = e & \text{if } ac = e, \\
  &ba = b^2a = ea = (bc)a = b(ca) = be = e & \text{if } ca = e, \\
  &ac = (ab)c = a(bc) = ae = e & \text{or } a & \text{if } ac = c, \\
  &ca = c(ba) = (cb)a = ea = e & \text{or } a & \text{if } ca = c.
\end{align*}
\]

Hence \( ac = a \) and \( ca = a \), that is, \( a \succ c \).
Since \( b \neq T \), it follows that \( a \in T \), hence \( e \gamma a \) by Lemma 9, and hence \( ac = (ab)c = a(bc) = ae = a \). If \( ca = a \), then \( a = ca = c^2a = ea = e \), contradicting. And if \( ca = c \), then \( c = ca = c(ab) = (ca)b - cb = e \), contradicting too. Hence \( ca = e \). \( c + a \) will be clear. Therefore \( a \geq c \).

(iii) We can prove in the similar way as (ii).

(iv) It follows that \( e = a \) by Lemma 14. Hence, if \( a \geq c \), then, by Lemma 13, we get \( a = e \). This is a contradiction. Therefore \( a \neq c \).

**Lemma 18.** Let \( b \sim c \), \( b + c \) and \( a \neq b \). Then \( b \geq a \) does not occur and

(i) \( b > a \) implies \( c > a \) or \( c \geq a \) if \( e \neq b \),

\[ c > a \]

(ii) \( b \geq a \) implies \( e = b \) and \( c > a \) or \( c \geq a \).

**Proof.** Let \( b \geq a \), then \( e \gamma b \) by Lemma 9, hence \( e \gamma b \). This contradicts with Lemma 4. Hence \( b \geq a \) does not occur.

(i) It follows that \( ca = (cb)a = c(ba) = cb = c \) and since if \( ac = a \), then \( b = ab = (ac)b = a(cb) = ac = a \), contradicting. Hence \( c > a \) or \( ca = c \) and \( ac = e \). By the way, letting \( ca = c \) and \( ac = e \), since \( a \neq c \) and \( a \neq e \) are clear, it holds that if \( c = e \), then \( c > a \), and if \( c \neq e \), then \( c \geq a \). Now, suppose that \( c \geq a \). Then it follows that \( eb = e \) and \( be = b \), hence \( e \gamma b \), and hence \( e = b \).

(ii) It holds that \( ca = (cb)a = c(ba) = cb = c \). And, if \( ac = a \), then \( e = ab = (ac)b = a(cb) = ac = a \) contradicting. So we get \( ac = e \) or \( c \). Hence \( c > a \) or \( ca = c \) and \( ac = e \). Letting \( ca = c \) and \( ac = e \), since \( a \neq e \) and \( a \neq c \) are clear, so it holds that if \( c = e \), then \( c > a \), and if \( c \neq e \), then \( c \geq a \). By Lemma 9, we have \( e \gamma b \), hence \( e = b \).

Similarly we have

**Lemma 19.** Let \( b \sim c \), \( b + c \) and \( a \neq b \). Then \( b \geq a \) does not occur and

(i) \( b > a \) implies \( c > a \) or \( c \geq a \) if \( e = b \),

\[ c > a \]

(ii) \( b \geq a \) implies \( e = b \) and \( c > a \) or \( c \geq a \).

Furthermore we have

**Lemma 20.** Let \( b \sim c \), \( b + c \) and \( a \neq b \). Then \( b \geq a \) implies \( a \neq c \).

**Proof.** Let \( b \sim c \) and \( b > a \). If \( ca = a \), then \( b = ba = (ca)b = c(ab) = cb = e \), contradicting, and if \( ac = a \), then \( b = ba = b(ac) = (ba)c = bc = e \), contradicting too. Hence \( a \neq c \). Next, let \( b \sim c \) and \( b \geq a \) \( \lbrack b \geq a \rbrack \). Since, then,

\[
\begin{align*}
    b &= ba = b(ca) = (bc)a = ea = e \\
    [b = ab = (ca)b = c(ab) = cb = e] \\
    b &= ba = b(ac) = (ba)c = bc = e \\
    [b = ab = (ac)b = a(cb) = ae = a or e]
\end{align*}
\]
we arrive at \( ca-e \) or \( c \) and \( ac=e \) of \( c \), hence \( a \neq c \).

3. The structure of \( A \).

Let \( \bar{S} = \{ S_a \}_{a \in A} \) be the set of all equivalence classes of \( S \) modulo \( \simeq \). Now we shall define \( \geq \) and \( \neq \) on \( S \) as follows:

\[
S_a \geq S_b \text{ means } S_a = S_b \text{ or } x > y \text{ for every } x \in S_a \text{ and every } y \in S_b,
\]

\[
S_a \neq S_b \text{ does } x \neq y \text{ for every } x \in S_a \text{ and every } y \in S_b.
\]

By \( S_a > S_b \) we denote that \( S_a \geq S_b \) and \( S_a \neq S_b \). Then we easily have

**Lemma 21.** \( \geq \) is a partial ordering on \( S \).

And we have that

**Lemma 22.** (i) If \( S_a(\alpha \neq \varepsilon) \) has at least one element \( x \) such that \( x > e \), then \( S_a > S_e \).

(ii) \( S_a \neq S_b \) and \( S_a > S_e \). Especially \( S_a > S_e \) if \( S_a \neq \varnothing \).

(iii) If \( S_a \neq \varnothing \), then for any \( \alpha \) of \( \Lambda \) one and only one of \( S_a > S_\alpha \), \( S_a = S_\alpha \) and \( S_a > S_\beta \),

holds.

(iv) \( S_a > S_\alpha \) implies \( S_a \geq S_\alpha \), and \( S_a > S_\beta \) implies \( S_a \geq S_\beta \).

(v) For any \( \alpha, \beta (\neq \varepsilon, \nu) \), one and only one of \( S_a > S_\alpha \), \( S_a = S_\beta \), \( S_\alpha > S_a \) and \( S_\alpha \neq S_\beta \) holds.

(vi) Unless \( S_a \neq S_b \), and \( S_b \neq S_a \), then one and only one of \( S_a > S_b \), \( S_a = S_b \) and \( S_b > S_a \),

holds.

(vii) If there exists \( \gamma (\neq \varepsilon) \in A \) such that \( S_\gamma \geq S_a \) and \( S_\gamma \geq S_b \), then one and only one of \( S_a > S_\gamma \), \( S_a = S_\gamma \) and \( S_\gamma > S_a \) holds.

**Proof.** (i) By Lemma 15, 16, 18, 19 and \( w > e > u \), \( u \in U \), we get \( S_a > S_e \).

(ii) \( S_a > S_e \) will be clear. If \( S_a > S_e \), for any \( y \in S_a \) and \( e \in S_a \), it follows that \( y > e \) and \( e > y \), contradicting. Of course \( S_a \neq S_a \). Hence \( S_a \neq S_a \). And if \( S_a \neq \varnothing \), then \( S_a = S_\varepsilon \), hence \( S_a > S_e \).

(iii) For any \( x \in S_a \), it follows that \( x > e \) or \( e \equiv x \) or \( e > x \). If \( e \equiv x \), then \( S_a = S_a \). And, by Lemma 15, 16, 17, 18, 19 and \( S_a \neq \varnothing \), we have that if \( x > e \), then \( S_a > S_e \), and, if \( x < e \), then \( S_a < S_e \).

(iv) Let \( S_a > S_e \). Then for any \( x \in S_a \) and any \( y \in S_a \), it follows that \( x > e \) and \( e > y \), hence \( x > y \), and hence \( S_a > S_e \). Next, let \( S_a > S_e \). Then for any \( x \in S_a \), it holds that \( e \equiv x \) or \( e > x \) or \( x > e \). If \( e \equiv x \), then \( S_a = S_a \), if \( e > x \), then, by Lemma 13, it follows that \( e > x > y \), \( y \in S_a \), implies \( e \equiv x \), contradicting, and if \( x > e \), then \( S_a > S_a \), by (i) in this lemma.

(v) Let any \( x \in S_a \) and any \( y \in S_b \). Then it follows that all of \( x \geq y \), \( x \geq y \), \( y \geq x \) and \( y \geq x \) do not occur. In fact, for example, supposing \( x \geq y \), we have that if \( x \notin T \), then \( x \in U \) by Lemma 9, hence \( S_a = S_a \), contradicting, and if \( x \notin T \), then \( e \equiv x \) by Lemma 9, hence \( S_a = S_a \), contradicting too. Now, if \( x \equiv y \), then \( S_a = S_b \), and if \( x \neq y \), then, by Lemma 6, 15, 16, 18, 19, it holds that \( S_a > S_b \) if \( x > y \), \( S_b > S_b \) if \( y > x \) and \( S_a \neq S_b \), if \( x \neq y \).

(vi) We may suppose that \( S_a > S_b \), without loss of generality. For any \( x \in S_a \) and
any \( y \in S_y \), if \( x \neq y \), then it follows that \( x > y \) or \( y > x \). Because, if \( x \geq y \), then \( e = x \) or \( e > x \), if \( y \geq x \), then \( e > x \), and if \( x \neq y \), then \( e > x \) and these are all contradictions. Therefore we get \( S_x = S_y \) if \( x = y \), \( S_x > S_y \) if \( x > y \) and \( S_y > S_x \) if \( y > x \).

(vii) Let any \( x \in S_x \), any \( y \in S_y \) and any \( z \in S_z \). Then it is easily shown that \( e > z \), \( x \in T \) and \( y \in T \), because if not so, we arrive at \( S_x \geq S_y \), contradicting. And, by Lemma 9, 14, we can prove that all of \( x \neq y \), \( x \geq y \), \( x > y \), \( y > x \) and \( y \geq x \) do not occur. Therefore we have \( x = y \) or \( x > y \) or \( y > x \). Thus the proof has been completed.

Define \( \geq \) (\( > \) or \( = \)) and \( \neq \) on \( A \) as follows:

\[ a > \beta \text{ means } S_x > S_y, \quad a = \beta \text{ does } S_x = S_y \text{ and } \alpha \neq \beta \text{ does } S_x \neq S_y. \]

Then it is clear that \( S \) is order-isomorphic onto \( A \) under a mapping \( S_x \rightarrow \alpha \). Therefore the above Lemma 22 can be rewritten as follows:

**Theorem 2.** \( A \) is a partially ordered set with respect to \( \geq \) which contains a definite element \( e \) and has the following properties:

(i) If \( w \in A \), then for any \( \alpha \in A \) one and only one of \( \alpha > e \), \( \alpha = e \) and \( e > \alpha \) holds.

(ii) \( \alpha > e \) implies \( \alpha > \beta \), and \( \alpha > \beta \) implies \( \alpha \geq e \).

(iii) For any \( \alpha \), \( \beta \neq e, \nu \) of \( A \) one and only one of \( \alpha > \beta \), \( \alpha = \beta \), \( \beta > \alpha \) and \( \alpha \neq \beta \) holds.

(iv) Unless \( \alpha \neq e \) and \( \beta \neq e \), then one and only one of \( \alpha > \beta \), \( \alpha = \beta \) and \( \beta > \alpha \) holds.

(v) If there exists \( r ( \neq e, \neq \nu ) \in A \) such that \( r \leq \alpha \) and \( r \geq \beta \), then one and only one of \( \alpha > \beta \), \( \alpha = \beta \) and \( \beta > \alpha \) holds.

4. **Subsets of \( S \) defined for the element of \( A \).**

Take any \( \alpha \) \((\neq e)\) of \( A \). For a fixed element \( x \in S_x \), we put

\[ S^c(a) = \{ y \in S_y : y > x \}, \quad S_1(a) = \{ y \in S_y : y \geq x \}, \quad S_2(a) = \{ y \in S_y : y \geq x \}, \quad S_3(a) = \{ y \in S_y : y > x \}, \quad S_4(a) = \{ z \in S_z : z > x \}, \quad S_5(a) = \{ z \in S_z : z \geq x \}. \]

Then \( S_1(a), S_2(a) \) and \( S_3(a) \) are defined for all \( \alpha \neq e \in A \) and \( S_4(a), S_5(a) \) and \( S_6(a) \) are done for all \( \alpha \in A \) such that \( \alpha \neq e \) and \( \alpha \neq \nu \). And it will be clear that these all subsets are determined uniquely by \( \alpha \) and are mutually disjoint. Furthermore we have

**Theorem 3.** (i) \( S_6(a) = \) \( e \) for every \( \alpha \neq e \in A \). Specially \( S_6(\nu) = \{ e \} \).

(ii) For every \( \alpha \neq e \in A \) it holds that

\[ S_x = \begin{cases} S_6(a) + S_1(a) & \text{if } x \text{ is a left zero semigroup}, \\ S_1(a) + S_3(a) & \text{if } x \text{ is a right zero semigroup}, \end{cases} \]

and for every \( \alpha \neq e, \neq \nu \in A \)
\[ S_\tau = S_x(\alpha) + S_x'(\alpha) + S_y(\alpha) + \tilde{S}_x(\alpha). \]

(iii) For every \( x \in S_x(\alpha \neq \varnothing) \) it follows that \( y \geq x \) for every \( y \in S_x(\alpha) \), \( y \geq x \) for every \( y \in S_x'(\alpha) \), and \( y \geq x \) for every \( y \in S_y(\alpha) \). And for every \( x \in S_x(\alpha \neq \varnothing, \neq \varnothing) \) it holds that \( z \geq x \) for every \( z \in S_x(\alpha) \), \( z \geq x \) for every \( z \in S_x'(\alpha) \), and \( z \geq x \) for every \( z \in \tilde{S}_x(\alpha) \).

(iv) For \( \alpha(\neq \varnothing, \neq \varnothing) \) and \( \beta(\neq \varnothing, \neq \varnothing) \) it follows that if \( \alpha > \beta \), then

1) \( S_x(\alpha) \subseteq S_x(\beta) \)
2) \( S_x(\alpha) \subseteq S_x(\beta) \)
3) \( S_x'(\alpha) \subseteq S_x'(\beta) \)
4) \( S_y(\alpha) \subseteq S_y(\beta) \)

and if \( \alpha \equiv \beta \), then

5) \( S_x(\alpha) \subseteq \tilde{S}_x(\beta) \)
6) \( S_x'(\alpha) \subseteq \tilde{S}_x(\beta) + S_x'(\beta) \)
7) \( S_y(\alpha) \subseteq \tilde{S}_y(\beta) + S_y'(\beta) \)
8) \( S_y(\beta) \subseteq \tilde{S}_y(\alpha) + S_y'(\alpha) \)

Proof. (i) Let any \( x \in S_x \). Then, by Lemma 18, 19, we get \( x \neq \varnothing \), hence \( e \geq x \), and hence \( S_x(\alpha) \neq \{ \} \). Next, let \( y \in S_y \) and \( z \in S_z \), then \( e = z \) and \( e > y \). Therefore if \( z \in S_y(\nu) \), then

\[ z = \begin{cases} 
  yz = y^2z - ez = e & \text{if } S_y \text{ is a left zero semigroup,} \\
  zy = zy^2 - ze = e & \text{if } S_y \text{ is a right zero semigroup},
\end{cases} \]

hence \( S_y(\nu) = \{ e \} \).

(ii) Let any \( x \in S_x \), any \( y \in S_y \), and any \( z \in S_z \). Since \( e > x \) for \( x \) and \( y \), by Lemma 15, 16, 18, 19, it follows that any one of \( y > x \), \( y \geq x \) and \( y \geq x \) holds. Therefore \( x \in \bar{S}_x(\alpha) + S_y(\alpha) \) or \( x \in \bar{S}_x(\alpha) + S_y'(\alpha) \), so we get

\[ S_x = \bar{S}_x(\alpha) + S_y(\alpha) \quad \text{if } S_y \text{ is a left zero semigroup,} \]
\[ S_x = \bar{S}_x(\alpha) + S_y'(\alpha) \quad \text{if } S_y \text{ is a right zero semigroup}. \]

Next, suppose that \( \alpha \neq \nu \). It will be clear that \( x \neq \varnothing \) by Lemma 13. Hence any one of \( z > x \), \( z \geq x \), \( z \geq x \) and \( z \neq x \) holds. Therefore \( z \in \bar{S}_x(\alpha) + S_x'(\alpha) + S_y(\alpha) + \tilde{S}_x(\alpha) \), hence \( \bar{S}_x = \bar{S}_x(\alpha) + S_x'(\alpha) + \tilde{S}_x(\alpha) \).

(iii) These are obvious.

(iv) Let any \( x \in S_x \), any \( y \in S_y \).

1) The case where \( \alpha > \beta \). If \( a \in \bar{S}_x(\alpha) \), then \( a > x \), while \( x > y \), so we get \( a > y \). Hence \( a \in \bar{S}_x(\beta) \) and hence \( \bar{S}_x(\alpha) \subseteq \bar{S}_x(\beta) \). Similarly we get \( \bar{S}_y(\alpha) \subseteq \bar{S}_y(\beta) \). If \( a \in S_y'(\alpha) [a \in S_y'(\alpha)] \),
then \(a \geq x[a \geq x]\), while \(x > y\), hence \(a \geq y\) or \(a > y\) \([a \geq y\) or \(a > y\)] by Lemma 11. Hence
we have that \(S_x'(\alpha) \subseteq S_x(\beta) + S_x'(\beta)[S_x'(\alpha) \subseteq \bar{S}_x(\beta) + S_x'(\beta)]\).

2) The case where \(a \notin \beta\). If \(a \in S_x(\alpha)\), then \(a > x\), while \(x \notin y\), hence \(ya = y(xa)\) =
\((yx)a = a = ay = -a = a = a(\beta) = (ax) y = ay\). Since \(a \notin e\), \(y \notin y\) and \(a \notin y\), we get \(a \notin y\),
hence \(S_x(\alpha) \subseteq \bar{S}_x(\beta)\). If \(a \in S_x'(\alpha)\), then \(a \geq x\), while \(x \notin y\), hence \(ay = (ax) y = a (xy) = ae = e\) and \(ya = a\) or \(e\). In fact, if \(ya = y\), then \(y = ya = y(ay) = (ya)x = xy = e\), contradicting. Since \(a \notin e\), \(y \notin e\), \(a \notin y\), \(y \in T\), it follows that \(a \geq y\) or \(a \notin y\), hence \(S_x'(\alpha) \subseteq \bar{S}_x(\beta) + S_x'(\beta)\).
Similarly we have \(S_x'(\alpha) \subseteq \bar{S}_x(\beta) + S_x'(\beta)\).

5. The structure of \(G = \sum S_x\).

According to Tamura [6], a \(\beta\)-semigroup was either
i) a zero semigroup defined by \(xy = e\) for all \(x, y\)
or ii) a semigroup which contains \(w \neq e\) and which is defined by \(wx = xw = w\) if \(x \neq w\);
\(xy = uy = e\) if \(x \neq w\), \(y \neq w\).

For convenience' sake we shall call i) and ii) a \(\beta_1\)-semigroup and a \(\beta_2\)-semigroup
respectively. By the way, by (i), (ii) and (iii) of Theorem 3, we obtain that \(T = S_x +
S_x\) is a semigroup, which shall be called a \(\bar{\beta}_1\)-semigroup, defined by \(xy = x[yx = x]\) for \(x
\in S_x\), \(y \in T\) and \(x'y = e[x'y = e]\) for \(x' \in S_x\), \(y \in T\) if \(S_x\) is a left [right] zero semigroup,
and that the difference semigroup of \(T\) modulo \(S_x\) in Rees' sense [2] is a \(\beta_1\)-semigroup.

Lemma 23. Given a non-empty left or right zero semigroup \(A\) and a \(\beta_1\)-semigroup \(B\) which is disjoint from \(A\), and given a mapping \(\sigma\) of the set \(B\) of all non-zero elements
of \(B\) into \(A\) such that

\[\sigma : x \to x_0(\text{any fixed element of } A),\]

we can construct uniquely a \(\bar{\beta}_1\)-semigroup \(C\) such that \(C\) is the union of \(A\) and \(B\)
and the difference semigroup of \(C\) modulo \(A\) is isomorphic onto \(B\).

Proof. By (1) we denote the binary operation defined on \(A\). Define a binary
operation (\(\circ\)) on \(C = A \cup B\) as follows:

\[
x \circ y = \begin{cases} x \cdot y & \text{if } x \in A, y \in A, \\ x \cdot y & \text{if } x \in A, y \in B, \\ x \cdot y & \text{if } x \in B, y \in A, \\ x \cdot y & \text{if } x \in B, y \in B. \end{cases}
\]

Then we can prove easily that this lemma is true.

Thus we can determine the structure of \(G = \sum S_x\), that is

Theorem 4. \(G\) is either a \(\bar{\beta}_1\)-semigroup or a \(\beta_2\)-semigroup.

Proof. If \(S_x \neq \emptyset\), then \(S_x = \{e\}\), hence \(w > e > u\) for any \(u \in S_x\) and \(u \sim v\) for every
\[u, v \in S_. \text{ Hence } G \text{ is a } \beta_1\text{-semigroup. And if } S_0=\Box, \text{ then } G \text{ is clearly } \overline{\beta_1}\text{-semigroup.}

We shall call } G \text{ a } \bar{\beta}\text{-semigroup. Here, we note that } S_., S, \text{ and } S_\circ \text{ of } G \text{ can be written as follows:}

\begin{align*}
S_., &= \{x \in G; x^2 \neq x, x^3 = x\}, \\
S_\circ &= \{x \in G; x^2 = x\}, \\
S_0 &= \{x \in G; x^2 \neq x, x^3 \neq x\}
\end{align*}

and that } e, \text{ a definite element of } G, \text{ can be determined as}

\begin{align*}
\text{any fixed one element of } S, & \quad \text{if } S_\circ = G, \\
x^2, x \in S_\circ + S_0 & \quad \text{if } S_\circ \neq G.
\end{align*}

6. The structure theorem of } S.

Combining the Theorem 1, 2, 3 and 4, we can establish the following theorem.

**Theorem 5.** In order that a semigroup } S \text{ is a } \beta^*-\text{semigroup, it is necessary and sufficient that } S \text{ is uniquely expressible as a partially ordered set } A = \Delta_1 \cup \Delta_2 \cup \Delta_3 \text{ satisfying Theorem 2 of maximal left zero subsemigroups } S_, \lambda \in \Delta_1, \text{ maximal right zero subsemigroups } S_\circ, \mu \in \Delta_3, \text{ and a non-empty maximal } \bar{\beta}\text{-semigroup } G = \sum_{\lambda \in \Delta_1} S_, \Delta_2 = \{v, \varepsilon, \nu\}, \text{ which has mutually disjoint and uniquely determined subsets } S(\alpha), S_\circ(\alpha) \text{ (or } S_\circ(\alpha)) \text{, } S(S, \alpha), S(\alpha), S_\circ(S, \alpha), S(\alpha) \text{ and } S_\circ(\alpha) \text{ for all } \alpha(\neq \varepsilon, \neq \nu) \text{ of } A \text{ satisfying Theorem 3.}

**Proof.** We shall prove the sufficiency. Take any } x, y \in S \text{ and let } x \in S_., y \in S_\circ.

Then we can verify that } xy \text{ and } yx \text{ are equal to } e \text{ or } x \text{ or } y \text{ in the following each cases:}

\begin{itemize}
  \item[(i)] \(\alpha = \beta\),
  \item[(ii)] \(\alpha \neq \beta\) \text{ and } 1) \(\alpha, \beta \neq \varepsilon, \nu\),
  \item[2)] \(\alpha = \varepsilon, \beta \neq \nu\),
  \item[3)] \(\alpha = \nu, \beta \neq \varepsilon\),
  \item[4)] \(\alpha = \varepsilon, \beta = \nu\).
\end{itemize}

Therefore } S \text{ is a } \beta^*-\text{semigroup.}

**Corollary 1.** A } \beta^*-\text{semigroup } S \text{ is a } \beta\text{-semigroup if and only if } S \text{ has exactly one idempotent element.}

**Proof.** If } S \text{ has exactly one idempotent, then it follows that } S_\circ = S_\circ + |e| + S_. \text{ Hence } S \text{ is a } \beta\text{-semigroup. The converse is clear.}

**Corollary 2.** A } \beta^*-\text{semigroup } S \text{ is a Rédei's semigroup if and only if } S \text{ is an idempotent semigroup and } A \text{ is a linearly ordered set.}

**Proof.** If } S \text{ is an idempotent and } A \text{ is a linearly ordered set, then it follows that } S_\circ = \sum_{\alpha \in \Delta_1} S_\circ + \sum_{\beta \in \Delta_2} S_\circ + S_\circ \text{ and for any } \alpha, \beta \in A, \text{ any one of } \alpha > \beta, \alpha = \beta, \beta > \alpha \text{ holds. Hence for any } x(y \in S_\circ), y(\in S_\circ) \text{ of } S, \text{ any one of } x > y, x \approx y \text{ and } y > x \text{ holds, and hence } xy = x \text{ or } y. \text{ Therefore } S \text{ is a Rédei's semigroup. The converse is evident.}

**Corollary 3.** A } \beta^*-\text{semigroup } S \text{ is a left or right zero semigroup if and only if } S \text{ does not contain elements which commute with each other at all.}
Proof. If $A \ni \alpha (\neq \varepsilon)$, then we can take $x \in S_{\alpha}$ and we get $x e \neq e x$, hence $e x \neq e$. This contradicts with the assumption. Hence $A = \{e\}$. Therefore $S$ is a left or right zero semigroup. The converse is obvious.

7. Compositions of $\beta^s$-semigroups.

Suppose that there are given mutually disjoint systems $\{S_{\lambda}; \lambda \in D_1\}$ of mutually disjoint left zero semigroups, $\{S_{\mu}; \mu \in D_2\}$ of mutually disjoint right zero semigroups and a non-empty $\tilde{\beta}$-semigroup $G = \sum_{s \in D_1} S_s$, where $D = \{0, \varepsilon, \nu\}$ and $S_0 = \{x \in G; x^2 = x, x^3 = x\}$, $S_{\varepsilon} = \{x \in G; x^2 = x\}$ and $S_{\nu} = \{x \in G; x^2 \neq x, x^3 \neq x\}$, and that the suffix set $A = D_1 \cup D_2 \cup D_0$ is a partially ordered set with respect to $\succeq$ satisfying Theorem 2 and for all $\alpha (\neq \varepsilon$, $\neq \nu)$ of $A$ mutually disjoint subsets $S_{\alpha}, S_{\prime}(\alpha)$ (or $S_{\prime}(\alpha)$), $S_{\beta}(\alpha), S_{\prime}(\alpha)$ and $S_{\nu}(\alpha)$ of $G$ satisfying Theorem 3 are determined uniquely. Then, put $S = \sum_{\alpha \in D_1} S_{\alpha} + \sum_{\nu \in D_2} S_{\nu} + \sum_{s \in D_0} S_s$ and define $x y, x \in S_{\alpha}$ and $y \in S_{\beta}$ as follows:

(1) The case $\alpha = \beta$.

$$x y = \begin{cases} y & \text{if } \alpha \neq \beta, \\ y & \text{if } \beta \neq \alpha, \\ e (\text{definite element of } G) & \text{if } \alpha \not\equiv \beta. \end{cases}$$

(2) The case $\alpha \neq \beta$ and

a) $\alpha \in \{\varepsilon, \nu\}, \beta \in \{\varepsilon, \nu\}$.

$$x y = \begin{cases} x & \text{if } \alpha > \beta, \\ y & \text{if } \beta > \alpha, \\ e & \text{(definite element of } G) \text{ if } \alpha \not\equiv \beta. \end{cases}$$

b) $\alpha = \varepsilon$, $\beta \not\equiv \nu$.

$$x y = y x \quad \text{if } \beta \not\equiv \varepsilon, \quad x y = x = x y \quad \text{if } \beta \not\equiv \varepsilon \text{ and } x \in S_{\tilde{\varepsilon}}(\beta), \quad x y = e, y x = e \quad \text{if } \beta \not\equiv \varepsilon \text{ and } x \in S_{\tilde{\varepsilon}}(\beta).$$

c) $\alpha = \nu$, $\beta \not\equiv \varepsilon$.

$$x y = y = y x \quad \text{if } \beta \not\equiv \varepsilon, \quad x y = x = x y \quad \text{if } \beta \not\equiv \varepsilon \text{ and } x \in S_{\tilde{\nu}}(\beta), \quad x y = e, y x = e \quad \text{if } \beta \not\equiv \varepsilon \text{ and } x \in S_{\tilde{\nu}}(\beta).$$
Then we can prove that $S$ forms a $\beta^*$-semigroup with respect to the above binary operation. That is

**Theorem 6.** Any $\beta^*$-semigroup $S$ is constructed in the above mentioned way.

**Proof.** If the new binary operation is associative, then $S$ is a $\beta^*$-semigroup since $xy=e$ or $x$ or $y$ for every $x, y \in S$. Now we shall prove that $x(yz)=(xy)z$ for every $x, y, z \in S$. Let $x \in S$, $y \in S_j$ and $z \in S_j$. There are the following cases:

\[ a = \beta = \gamma \quad \text{and} \]
\[ (1.1) \quad a \in \Delta_j \cup \Delta_n, \quad (1.2) \quad a \in \Delta_n. \]

\[ a = \beta, \quad \alpha \neq \gamma \quad \text{and} \]
\[ (2.1) \quad \alpha = \epsilon, \quad \gamma \neq \nu, \quad (2.2) \quad \epsilon = \nu, \quad \gamma \neq \epsilon, \quad (2.3) \quad \alpha \neq \nu, \quad \gamma = \epsilon, \quad (2.4) \quad \alpha \neq \epsilon, \quad \gamma = \nu, \quad (2.5) \quad \alpha = \epsilon, \quad \gamma = \nu, \quad (2.6) \quad \alpha = \nu, \quad \gamma = \epsilon, \quad (2.7) \quad \alpha, \gamma \in \{\epsilon, \nu\}. \]

\[ a = \gamma, \quad \beta \neq \gamma \quad \text{and} \]
\[ (3.1) \quad \beta = \epsilon, \quad \gamma \neq \nu, \quad (3.2) \quad \beta = \nu, \quad \gamma \neq \epsilon, \quad (3.3) \quad \beta \neq \nu, \quad \gamma = \epsilon, \quad (3.4) \quad \beta \neq \epsilon, \quad \gamma = \nu, \quad (3.5) \quad \beta = \epsilon, \quad \gamma = \nu, \quad (3.6) \quad \beta = \nu, \quad \gamma = \epsilon, \quad (3.7) \quad \beta, \gamma \in \{\epsilon, \nu\}. \]

\[ \beta = \gamma, \quad \alpha \neq \beta \quad \text{and} \]
\[ (4.1) \quad \alpha = \epsilon, \quad \beta \neq \nu, \quad (4.2) \quad \alpha = \nu, \quad \beta \neq \epsilon, \quad (4.3) \quad \alpha \neq \nu, \quad \beta = \epsilon, \quad (4.4) \quad \alpha \neq \epsilon, \quad \beta = \nu, \quad (4.5) \quad \alpha = \epsilon, \quad \beta = \nu, \quad (4.6) \quad \alpha = \nu, \quad \beta = \epsilon, \quad (4.7) \quad \alpha, \beta \in \{\epsilon, \nu\}. \]

\[ a = \beta = \gamma \quad \text{and} \]
\[ (5.1) \quad \alpha = \epsilon, \quad \beta \neq \nu, \quad (5.2) \quad \alpha \neq \nu, \quad \beta = \epsilon, \quad (5.3) \quad \alpha \neq \nu, \quad \beta \neq \nu, \quad (5.4) \quad \alpha \neq \nu, \quad \beta = \epsilon, \quad (5.5) \quad \alpha = \nu, \quad \beta \neq \epsilon, \quad (5.6) \quad \alpha = \epsilon, \quad \beta \neq \epsilon, \quad (5.7) \quad \alpha = \epsilon, \quad \beta = \nu, \quad (5.8) \quad \alpha = \nu, \quad (5.9) \quad \beta = \epsilon, \quad (5.10) \quad \alpha = \nu, \quad \beta = \epsilon, \quad (5.11) \quad \alpha = \nu, \quad \gamma = \epsilon, \quad (5.12) \quad \beta = \nu, \quad \gamma = \epsilon, \quad (5.13) \quad \alpha, \beta, \gamma \in \{\epsilon, \nu\}. \]

Here, we shall prove only the cases (2.1), (2.7), (5.1), (5.7) and (5.13).

(1') The case where (2.1). If $\gamma > \epsilon$, then $x(yz)=z=(xy)z$. And if $\gamma \neq \epsilon$, then, by (ii) of Theorem 3,

\[ x(yz) = \begin{cases} 
xy = x \cdot xz = (xy)z & \text{if } x \in S_1(\gamma) \text{ or } S_1(\gamma), \\
yz = (xy)z & \text{if } x \in S_1(\gamma). 
\end{cases} \]

(2') The case where (2.7). By (iii) or Theorem 2, possible cases are $\alpha > \gamma$, $\gamma > \alpha$ and $\alpha \neq \gamma$. If $\alpha > \gamma$, then, by (iii) of Theorem 3, $x(yz)=xy=(xy)z$. If $\gamma > \alpha$, then $x(yz)=z=(xy)z$. And if $\alpha \neq \gamma$, then, by (iv) of Theorem 2, it holds $\alpha \neq \epsilon$, hence, by (i) of Theorem 3, we get $e \in S_1(\alpha)$, and hence $x(yz)=e=(xy)z$.

(3') The case where (5.1).

1) The case $\beta$ and $\gamma > \epsilon$. By (iv) of Theorem 2, for $\beta$ and $\gamma$ there are two cases
\( \beta > \gamma, \gamma > \beta \). Hence we get

\[
x(yz) = \begin{cases} 
xy - y = yz - (xy)z & \text{if } \beta > \gamma, \\
xz - z = yz - (xy)z & \text{if } \gamma > \beta.
\end{cases}
\]

2) The case \( \beta > \varepsilon \) and \( \gamma \equiv \varepsilon \). If \( \gamma > \beta \), then it follows that \( \gamma > \varepsilon \), contradicting. Hence \( \gamma \equiv \beta \), and hence we get \( \beta > \gamma \) by (iv) of Theorem 2. Hence \( x(yz) = xy = yz = (xy)z \).

3) The case \( \beta \equiv \varepsilon \) and \( \gamma > \varepsilon \). If \( \beta > \gamma \), then \( \beta > \varepsilon \), contradicting. Hence \( \beta \equiv \gamma \), and hence \( \gamma > \beta \). Therefore \( x(yz) = xz = z \). On the other hand we have

\[
x(yz) = \begin{cases} 
xz & \text{if } x \in S_1(\beta) \text{ or } S_1(\beta), \\
ez & \text{if } x \in S_1(\beta).
\end{cases}
\]

4) The case \( \beta \equiv \varepsilon \) and \( \gamma \equiv \varepsilon \). By (iii) of Theorem 2, for \( \beta \) and \( \gamma \) there are three cases \( \beta > \gamma \), \( \gamma > \beta \) and \( \beta \equiv \gamma \). Suppose that \( \beta > \gamma \). If \( x \in S_1(\beta) \), then, by (iv) of Theorem 3, it follows that \( x \in S_1(\gamma) \), hence \( x(yz) = xy = xz = (xy)z \). If \( x \in S_1(\beta) \), then, by (ii) of Theorem 3, we get \( x \in S_1(\gamma) \) or \( S_1(\gamma) \), hence \( x(yz) = xy = xz = (xy)z \). And if \( x \in S_1(\beta) \), then \( x(yz) = xe = e = ez = (xy)z \). Supposing \( \gamma > \beta \), similarly, we have \( x(yz) = (xy)z \). And suppose that \( \beta \equiv \gamma \). If \( x \in S_1(\beta) \) or \( S_1(\beta) \), then \( x \in S_1(\gamma) \) or \( S_1(\gamma) \), hence \( x(yz) = xe = xz = (xy)z \), and if \( x \in S_1(\beta) \), then \( x(yz) = xe = e = ez = (xy)z \).

(4') The case where \( (5,7) \).

1) The case \( \gamma > \varepsilon \). It follows that \( \gamma > \beta \), hence \( x(yz) = xz = z \) and

\[
(xyz) = \begin{cases} 
xz & \text{if } x \in S_1(\mu) \text{ or } S_1(\mu), \\
ez & \text{if } x \in S_1(\mu).
\end{cases}
\]

2) The case \( \gamma \equiv \varepsilon \). If \( \gamma > \mu \), then \( \gamma > \varepsilon \), contradicting. Hence \( \gamma \equiv \mu \equiv \mu \). Hence, if \( x \in S_1(\beta) \), then, by (i) of Theorem 3, we get \( x = e \), hence \( (xy)z = ez = e \) and

\[
x(yz) = \begin{cases} 
xz & \text{if } y \in S_1(\mu) \text{ or } S_1(\mu), \\
ez & \text{if } y \in S_1(\mu) \text{ or } S_1(\mu).
\end{cases}
\]

If \( x \in S_1(\beta) \), then \( x \in S_1(\gamma) \) or \( S_1(\gamma) \), hence

\[
x(yz) = \begin{cases} 
xy = xz = (xy)z & \text{if } y \in S_1(\mu) \text{ or } S_1(\mu), \\
ex = xz = (xy)z & \text{if } y \in S_1(\mu) \text{ or } S_1(\mu).
\end{cases}
\]

And if \( x \in S_1(\beta) \), then \( x \in S_1(\gamma) \) or \( S_1(\gamma) \), hence

\[
x(yz) = \begin{cases} 
xy = ez = (xy)z & \text{if } y \in S_1(\mu) \text{ or } S_1(\mu), \\
xe = ez = (xy)z & \text{if } y \in S_1(\mu) \text{ or } S_1(\mu).
\end{cases}
\]
(5') The case where (5.13).

From (iii) of Theorem 2, we have the following subcases:

(1) \( \alpha > \beta, \gamma > \beta, \alpha \nleq \gamma \), (2) \( \alpha > \beta, \beta \nleq \gamma, \alpha > \gamma \), (3) \( \alpha > \beta, \beta \nleq \gamma, \alpha \nleq \gamma \),
(4) \( \beta > \alpha, \beta > \gamma, \alpha \nleq \gamma \), (5) \( \beta > \alpha, \beta \nleq \gamma, \gamma > \alpha \), (6) \( \beta > \alpha, \beta \nleq \gamma, \alpha \nleq \gamma \),
(7) \( \alpha \nleq \beta, \beta > \gamma, \alpha > \gamma \), (8) \( \alpha \nleq \beta, \beta > \gamma, \alpha \nleq \gamma \), (9) \( \alpha \nleq \beta, \gamma > \beta, \gamma > \alpha \),
(10) \( \alpha \nleq \beta, \gamma > \beta, \gamma \nleq \gamma \), (11) \( \alpha \nleq \beta, \beta \nleq \gamma, \gamma > \gamma \), (12) \( \alpha \nleq \beta, \beta \nleq \gamma, \gamma > \alpha \),
(13) \( \alpha \nleq \beta, \beta \nleq \gamma, \alpha \nleq \gamma \), (14) \( \alpha > \beta, \beta > \gamma \), (15) \( \alpha > \beta, \gamma > \beta \),
(16) \( \beta > \alpha, \alpha > \gamma \), (17) \( \gamma > \alpha, \beta > \gamma \), (18) \( \alpha > \beta, \gamma > \alpha \), (19) \( \beta > \alpha, \gamma > \beta \).

We prove here (2), (3) and (5) only.

1) The case (2). If \( \alpha \nleq \varepsilon \), then, by (v) of Theorem 2, for \( \beta \) and \( \gamma \) one and only one of \( \beta > \gamma \) and \( \gamma > \beta \) holds, contradicting to \( \beta \nleq \gamma \). Hence \( \alpha \nleq \varepsilon \), and so \( x(yz) = xe = x = xz = (xy)z \).

2) The case (3). If \( \alpha > \varepsilon \), then, by (iv) of Theorem 2, it follows that for \( \alpha \) and \( \gamma \) one and only one of \( \alpha > \gamma \) and \( \gamma > \alpha \) holds, contradicting. Hence \( \alpha \nleq \varepsilon \), and hence \( e \in S_\alpha (\alpha) \). Therefore \( x(yz) = xe = e = yz = (xy)z \).

3) The case (5). It follows that \( \beta \nleq \varepsilon \), hence \( \alpha \nleq \varepsilon \). Hence \( e \in S_{\alpha}(\alpha) \), so that \( x(yz) = xe = e = yz = (xy)z \). Therefore, we have that \( x(yz) = (xy)z \) for all \( x, y, z \) of \( S \). Thus the proof has been finished.

8. The isomorphism problem of \( \beta \)-semigroups.

Let \( S = \sum_{a \in A} S_a \) and \( S' = \sum_{a' \in A'} S'_{a'} \) be two \( \beta \)-semigroups composed by the above mentioned way. And let \( G = S + S + S \) and \( G' = S + S + S \) be the \( \beta \)-semigroups of \( S \) and \( S' \) respectively and let \( S_\alpha(\alpha), S_\alpha(\alpha), S_\alpha(\alpha), S_\alpha(\alpha), S_\alpha(\alpha), S_\alpha(\alpha), S_\alpha(\alpha), S_\alpha(\alpha), S_\alpha(\alpha), S_\alpha(\alpha) \) and \( S_\alpha(\alpha), S_\alpha(\alpha) \) be the subsets of \( G \) and \( G' \) which are defined for all \( \alpha (\neq \varepsilon, \neq \nu) \) of \( A \) and \( \alpha' (\neq \varepsilon, \neq \nu') \) of \( A' \) respectively. Suppose that \( S \) is isomorphic onto \( S' \) under \( \sigma \). Since \( (aa)(bb) = aa \) if and only if \( ab = c \) for \( a, b, c \) of \( S \), it follows that \( Go = G' \) and \( \sigma = \sigma' \), where \( e, e' \) are the definite elements of \( G \), \( G' \) respectively. Consider a mapping \( \varphi \) of \( A \) to \( A' : \alpha \rightarrow \alpha' (\alpha' \) is an element of \( A' \) such that \( \alpha \sigma \in S_{\alpha} \). Then we have immediately that \( \varphi \) is onto and one-to-one. If \( \beta \geq \alpha \), then, since for any \( x' \in S_{\alpha} \) and for any \( y' \in S_{\beta} \) there exist \( x \in S \) and \( y \in S \) such that \( x' = x \) and \( y' = y \), it follows that \( x \leq y \), hence \( x = x' = y = y \), and hence \( \alpha \varphi = \beta \beta \varphi \). Similarly we get \( \alpha \varphi = \beta \varphi \) implies \( \alpha \leq \beta \). Hence \( \varphi \) is an order isomorphism of \( A \) onto \( A' \). Next, define mappings \( \varphi \) of \( S \) to \( S_{\alpha} \) \( \alpha \in A \cup A_\varepsilon \) and \( \varphi \) of \( G \) to \( G' \) as follows: for \( x \in S \),

\[
\begin{align*}
\varphi_a : x & = x a & \text{if } \alpha \in A_1 \cup A_2, \\
\varphi_0 : x & = x a & \text{if } \alpha \in A_0.
\end{align*}
\]

Then we get easily that \( \psi_a \) is an isomorphism of \( S_a \) onto \( S'_{\alpha} \), \( \alpha \in A_1 \cup A_2 \), and \( \psi_0 \) is an isomorphism of \( G \) onto \( G' \). Furthermore, we can prove that for all \( \alpha (\neq \varepsilon, \neq \nu) \) of \( A \)
the following equalities hold:

\[
\begin{align*}
(S(\alpha))\phi_0 &= S'_\alpha (\alpha \varphi), \\
(S'_\alpha (\alpha))\phi_0 &= S''_\alpha (\alpha \varphi), \\
(S'_\alpha (\alpha))\psi_0 &= S'_{\alpha'} (\alpha \varphi), \\
(S'_{\alpha'} (\alpha))\psi_0 &= S''_{\alpha'} (\alpha \varphi), \\
(S'_{\alpha'} (\alpha))\psi_0 &= S''_{\alpha'} (\alpha \varphi).
\end{align*}
\]

(\*)

Therefore we have

**Theorem 7.** $S = \sum_{\alpha \in \Delta} S_{\alpha}$ is isomorphic onto $S' = \sum_{\alpha' \in \Delta'} S'_{\alpha'}$ if and only if there are an order isomorphism $\varphi$ of $\Delta$ onto $\Delta'$, isomorphisms $\phi_\alpha$ of $S_{\alpha}$ onto $S'_{\alpha'}$ for all $\alpha \in \Delta \cup \Delta'$, and an isomorphism $\psi_\alpha$ of $G$ onto $G'$ satisfying the above condition (\*).

**Proof.** We shall prove the sufficiency. Define a mapping $\sigma$ of $S$ to $S'$ as follows: for $x \in S_{\alpha}$

\[
\sigma : x \rightarrow \begin{cases} 
\psi_{\alpha} & \text{if } \alpha \in \Delta \cup \Delta', \\
\psi_0 & \text{if } \alpha \in \Delta_0.
\end{cases}
\]

Then $\sigma$ is a one-to-one and onto. To prove that $\sigma$ is a homomorphism, take any $x \in S_{\alpha}$ and any $y \in S_{\beta}$.

1. The case where $\alpha = \beta$.

\[
(x\sigma)(y\sigma) = \begin{cases} 
(x\psi_{\alpha})(y\psi_{\alpha}) = (xy)\psi_{\alpha} = (xy)\sigma & \text{if } \alpha \in \Delta \cup \Delta', \\
(x\psi_{\alpha})(y\psi_{\alpha}) = (xy)\psi_{\alpha} = (xy)\sigma & \text{if } \alpha \in \Delta_0.
\end{cases}
\]

2. The case where $\alpha \neq \beta$.

1) The case $\alpha, \beta \in \Delta \cup \Delta'$.

\[
\begin{align*}
(x\sigma)(y\sigma) &= (x\psi_{\alpha})(y\psi_{\alpha}) = (xy)\psi_{\alpha} = (xy)\alpha \\
&= (xy)\sigma.
\end{align*}
\]

2) The case $\alpha, \beta \in \Delta_0$.

\[
(x\sigma)(y\sigma) = (x\psi_{\alpha})(y\psi_{\alpha}) = (xy)\psi_{\alpha} = (xy)\alpha.
\]

3) The case $\alpha \in \Delta \cup \Delta'$, $\beta = \epsilon$.

\[
\begin{align*}
(x\sigma)(y\sigma) &= (x\psi_{\alpha})(y\psi_{\alpha}) = (xy)\psi_{\alpha} = (xy)\alpha \\
&= (xy)\sigma.
\end{align*}
\]

Similary we have that $(y\sigma)(x\sigma) = (yx)\sigma$.

4) The case $\alpha \in \Delta \cup \Delta'$, $\beta = \nu$. 

Similarly we get \((xy)\sigma = (yx)\sigma\).

5) The case \(\alpha \in \mathcal{A} \cup \mathcal{A}'\), \(\beta = \omega\). It follows that any one of \(\alpha > \beta\), \(\beta > \alpha\) holds and

\[
(x\alpha)(y\sigma) = (x\alpha)(y\sigma) = \begin{cases} x\alpha = (xy)\sigma & \text{if } \alpha > \varepsilon, \\ y\sigma = (xy)\sigma & \text{if } \alpha \geq \varepsilon, \ y \in S_\alpha(a) + S'_\alpha(a), \\ e\alpha = e\sigma = (xy)\sigma & \text{if } \alpha \geq \varepsilon, \ y \in S'_\alpha(a) + \tilde{S}_\alpha(a). \end{cases}
\]

Similarly we get \((yx)(x\sigma) = (yx)\sigma\).

Therefore we have that \(\sigma\) is a homomorphism. Thus the proof has finished.

Similarly we can prove that

**Theorem 8.** \(S = \sum S_\alpha\) is anti-isomorphic onto \(S' = \sum S'_\alpha\) if and only if there are an order isomorphism \(\varphi\) of \(A\) onto \(A'\), anti-isomorphisms \(\varphi_\alpha\) of \(S_\alpha\) onto \(S'_\alpha\) for all \(\alpha \in \mathcal{A} \cup \mathcal{A}'\), and an anti-isomorphism \(\psi_\alpha\) of \(G\) onto \(G'\) satisfying the following conditions: for all \(\alpha \in \mathcal{A} \cup \mathcal{A}'\) of \(A\)

\[
(S, (a), \psi_\alpha = \tilde{S}_\alpha(a), (S'_\alpha(a), \psi_\alpha = S'_\alpha(a), (S', (a), \psi_\alpha = S_\alpha(a), (S'_\alpha(a), \psi_\alpha = S'_\alpha(a),
\]

Immediately we have that

**Corollary 4.** Let \(S = \sum S_\alpha\) and \(S' = \sum S'_\alpha\) be Rédei's semigroups, Then \(S\) is isomorphic [anti-isomorphic] onto \(S'\) if and only if there are an order isomorphism \(\varphi\) of \(A\) onto \(A'\) and isomorphisms [anti-isomorphisms] \(\varphi_\alpha\) of \(S_\alpha\) onto \(S'_\alpha\) for all \(\alpha \in \mathcal{A}\).

By Corollary 4, we can calculate the number \(f(n)\) of all the isomorphically and anti-isomorphically distinct Rédei's semigroups of order \(n\). That is, we have the following corollary:

**Corollary 5.** \(f(n)\) is given as follows:

\[
f(n) = \sum_{m=1}^{n} f(n; m),
\]

where

\[
f(n; m) = \begin{cases} 1 & \text{if } m = 1 \text{ or } n, \\ n-1 & \text{if } m = n-1, \\ \sum_{p=1}^{r} (mC_p \times n-m-1C_{p-1} \times 2^{p-1}), \ r = \min \{m, n-m\} \text{ if } 2 \leq m \leq n-2. \end{cases}
\]

**Proof.** We shall consider the number \(f(n; m)\) of all the isomorphically and anti-isomorphically distinct Rédei's semigroups of order \(n\) satisfying \(|\mathcal{A}| = m\) for fixed \(m(1 \leq m \leq n)\). It will be clear that \(f(n; 1) = f(n; n) = 1\) and \(f(n; n-1) = n-1\). So we may
consider only the case $2 \leq m \leq n - 2$. Find, first, the number of ways in which we can fill up the fixed $m$ places with $n$ elements. This is clearly equivalent to finding the number of ways in which we can fill up all the $m$ places with $m$ elements taken from the $n$ elements, and then can rearrange the remaining $n - m$ elements to the $p$ (where $1 \leq p \leq \min \{m, n - m\}$) places taken from the $m$ places. Thus the required number is $\sum C_p \times n - m - 1 C_{p-1}$. Next, for each of these, arrange a left or a right zero semigroup to the places containing two or more than two elements. Then the number of all such ways is $\sum C_p \times n - m - 1 C_{p-1} \times 2^p$. Considering anti-isomorphic ones, we have that $f(n; m) = \sum (m C_p \times n - m - 1 C_{p-1} \times 2^p)$, $r = \min \{n, n - m\}$. Of course $f(n) = \sum f(n; m)$.

Giving the values of $f(n)$, $n \leq 10$, they are $f(1) = 1$, $f(2) = 2$, $f(3) = 4$, $f(4) = 9$, $f(5) = 21$, $f(6) = 50$, $f(7) = 120$, $f(8) = 289$, $f(9) = 697$, $f(10) = 1682$.

Iwate University,
Morioka, JAPAN

References