THE FORMULATION OF D-AFFINE CONNECTION

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D-擬似接続の構成

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Recently H. Flanders [2, 3] has defined the affine connection by extending the exterior differentiation operator to the operator that acts on some linear spaces in place of exterior differential forms. It is known that there exists a divergence operator of skew-symmetric tensors on an orientable $C^\infty$-manifold $M$, and it is useful to define the homology group on $M$ if $M$ is paracompact. On the other hand, the exterior differentiation operator plays the important role of defining the de Rham cohomology group on $M$. If $M$ is orientable and paracompact, these groups are isomorphic. Like this the exterior differentiation operator and the divergence operator are mutually dual. So we arrive at the question if we can define an analogous notion of the affine connection by employing the divergence operator on $M$. The solution of the question is the purpose of the present paper.

1. Preliminaries. Let $M$ be a connected $C^\infty$-manifold and $\mathbb{F}$ the ring of all $C^\infty$-functions on $M$. For a point $x \in M$ we denote by $\Lambda(x) = \bigoplus \Lambda^p(x)$ (direct-sum) the graded algebra of $p$-tensors, by convention, $\Lambda^0(x) = \mathbb{T}$ the tangent space of $M$ at $x$ and $\Lambda^0(x) = \mathbb{R}$ (the field of real numbers). If we put $\Lambda(M) = \bigcup_{x \in M} \Lambda(x)$, this is a fibre bundle over $M$. Let $\nu$ be a $C^\infty$-section of the bundle. Hence $\nu(x) \in \Lambda(x)$. If, for all $x \in M$, $\nu(x) \in \Lambda^p(x)$ for some fixed $p$, $\nu$ is called a $p$-tensor field. We denote by $T_p$ the space of $p$-tensor fields with $T_0 = \mathbb{T}$ the space of tangent vector fields and $T_0 = \mathbb{F}$. If we put $T = \bigoplus T_p$, $T$ is a graded algebra over $\mathbb{F}$.

As before, for a point $x \in M$ we denote by $\Lambda^*(x) = \bigoplus \Lambda^q(x)$ the graded algebra of exterior forms on the tangent space at $x$. By convention $\Lambda^0(x) = \mathbb{T}$ (the dual space of $T_x(M)$) and $\Lambda^0(0) = \mathbb{F}$. If we put $\Lambda^*(M) = \bigcup_{x \in M} \Lambda^*(x)$, this is a fibre bundle over $M$, and its $C^\infty$-section $\omega$ is called an exterior polynomial. Hence we have $\omega(x) \in \Lambda^*(x)$. If, for all $x \in M$, $\omega(x) \in \Lambda^q(x)$ for some fixed $q$, $\omega$ is called an exterior differential form of degree $q$, or simply a $q$-form. We denote by $T^q$ the space of $q$-forms with $T^0 = \mathbb{T}$ the space of tangent covector fields and $T^0 = \mathbb{F}$. If we put $T^* = \bigoplus T^q$, $T^*$ is a graded algebra over $\mathbb{F}$.

We shall consider a tensor products

$T_p^q = T_p \otimes T^q$.

1) Numbers in brackets refer to the bibliography at the end of the paper.
These spaces may be considered from two points of view: either as tensor products of the spaces $T_p$ and $T^q$ over the ring $\mathcal{R}$, or as sections of the fibre bundle of all the elements of all $(\Lambda^p(x)) \otimes (\Lambda^q(x))$, there the latter tensor product is taken over $R$. This implies, passing to homogeneous components, the existence of an operator on $T_p^q \times T_q^p$ to $T_{p+q}^p$ given by the linearity and

$$ (v \wedge w) \otimes (\omega \wedge \eta) = (v \otimes \omega) \wedge (w \otimes \eta) $$

where $v \in T_p$, $w \in T_q$, $\omega \in T^q$ and $\eta \in T^p$. It is easy to see that this operator is associative and distributive, and obeys the following commutation rule

$$ X \wedge Y = (-1)^{pq} Y \wedge X $$

where $X \in T_p$ and $Y \in T_q$. Clearly $\Sigma \otimes T_p^q$ is a graded algebra over $\mathcal{R}$ and naturally identified with $T \otimes T^*$.

Now we assume that $M$ is orientable. Let $\rho$ be a positive scalar density on $M$. In a coordinate neighbourhood $U$ with a local coordinate system $x^1, \ldots, x^n$, a $p$-tensor $v$ can be written by

$$ v = v_{i_1, \ldots, i_p}(x^1, \ldots, x^n) e_{i_1} \wedge \cdots \wedge e_{i_p} \quad (i_1, \ldots, i_p = 1, \ldots, n) $$

where $e_{i_1} \ldots e_{i_p}$ is a component of the Kronecker's tensor. The operator $\ast$ is $\mathcal{R}$-linear and has an inverse.

For a $q$-form $\omega (q \geq 1)$ written by

$$ \omega = a_{i_1, \ldots, i_q} dx^{i_1} \wedge \cdots \wedge dx^{i_q}, $$

another $\mathcal{R}$-linear operator $\ast'$ is defined by

$$ *' \omega = (1/\rho) a_{i_1, \ldots, i_q} e^{i_1} \cdots e^{i_q} \wedge e_{i_1} \cdots e_{i_q}. $$

By convention for $p = 0$ and $q = 0$, $*1 = 1/\rho e_1 \wedge \cdots \wedge e_n$ and $*1 = (1/\rho) e_1 \wedge \cdots \wedge e_n$ respectively.

Again consider a $p$-tensor field, we operate $*$ and $*' \; \text{for} \; v \; \text{by} \; \text{succession, and we have}$

$$ (*)' v = (*) (v) = (-1)^{pq} v. $$

Similarly we have for a $q$-form

$$ (\ast \ast') \omega = (\ast \ast') (\omega) = (-1)^{pq} \omega. $$

Hence we have

$$ (*)' v = (-1)^{pq} v, \quad *' \omega = (-1)^{pq} \ast \omega. $$

2. The dual wedge product In §3, we shall define the divergence of a $p$-tensor field. A usual wedge product of two divergence-free $p$-tensor fields is not in general divergence-free. Let us now construct a new and naturally defined product of two divergence-free $p$-tensor fields which is divergence-free. This product will be called
a dual wedge product.

For a \( p \)-tensor field \( v \) and \( p' \)-tensor field \( w \), the dual wedge product is defined by

\[
\omega \odot \eta = \begin{cases} 
\star^{-1}(\star \omega \wedge \star \eta) & \text{for } p + p' \geq n, \\
0 & \text{for } p + p' < n.
\end{cases}
\]

Similarly for \( q \)-form \( \omega \) and \( q' \)-form \( \eta \), the dual wedge product is defined by

\[
\omega \odot \eta = \begin{cases} 
\star^{-1}(\star \omega \wedge \star \eta) & \text{for } q + q' \geq n, \\
0 & \text{for } q + q' < n.
\end{cases}
\]

It is clear that these products are associative and distributive.

Next we shall define the dual wedge product for the space \( \Sigma \oplus T^0_p \), in fact, there exists an operator on \( T^k_p \times T^q_{p'} \) to \( T^m_{n-\frac{(p+q)}{2}} \) given by the linearity and

\[
(v \odot w) \otimes (w \odot \eta) = (v \otimes \omega) \circ (w \otimes \eta),
\]

where \( v \in T_p, \ w \in T_{p'}, \ \omega \in T^q \) and \( \eta \in T^q' \). This operator is also associative and distributive, and obeys the following commutation rule

\[
X \circ Y = (-1)^{(n-q)(n-q')} Y \circ X,
\]

where \( X \in T^q_p \) and \( Y \in T^q_{p'} \).

By convention we shall introduce a \( \mathfrak{F} \)-linear operator \# by

\[
\# X = (\star \otimes \star X) = (\star v) \otimes (\star \omega),
\]

where \( X = v \otimes \omega \in T^q_p \). Hence we have

\[
X \circ Y = \#^{-1}(\# X \wedge \# Y),
\]

where \( X \in T^q_p \) and \( Y \in T^q_{p'} \).

3. The divergence of \( p \)-tensor fields. Let \( M \) be a connected \( n \)-dimensional orientable \( \mathcal{C}^\infty \)-manifold and \( v \) a \( p \)-tensor field on \( M \). Writing locally \( v \) by (1. 3), we define the divergence of a \( p \)-tensor \( v \), written by \( \partial v \), by

\[
\partial v = \frac{1}{\rho} \frac{\partial (\rho \varepsilon^{i_1 \ldots i_p})}{\partial x^j} e_{i_1} \wedge \ldots \wedge e_{i_p},
\]

where \( \rho \) is a positive scalar density on \( M \). An easy calculation shows \( \partial \rho = \partial \rho = 0 \).

Proposition 3.1. In the graded algebra \( T = \mathcal{S} \oplus T^p \) over \( \mathfrak{F} \) (see § 1), there exists a unique operator \( \partial \), called the divergence, satisfying the following conditions:

(1) \( \partial \) is \( R \)-linear,

(2) \( v \in T_p \) implies \( \partial v \in T_{p-1} \),

(3) \( \partial(v \odot w) = \partial v \odot w + (-1)^{p-q} v \odot \partial w \) for \( v \in T_p \),

(4) \( \partial \rho = 0 \).

In fact, for the operators \( \star \) and \( \star' \) defined in § 1, we have

\[
\partial \varepsilon = \# \partial \varepsilon = (-1)^{(n-q)} \partial \varepsilon
\]

for a \( p \)-tensor field and for a \( q \)-form respectively. Hence for \( v \in T_p \) and \( w \in T_q \), by
employing (3.2), we have
\[ \mathcal{A}(v \wedge w) = \mathcal{A}(\ast^{-1}(\ast v \wedge \ast w)) \]
\[ = \ast^{-1}(d(\ast v \wedge \ast w)) \]
\[ = \ast^{-1}(d(\ast v \wedge \ast w) + (-1)^{n-p}(\ast \nu \wedge \ast \partial \nu)) \]
\[ = \ast^{-1}(\ast(\partial \nu) \wedge \ast w) + (-1)^{n-p-1}(\ast v \wedge \ast(\partial \nu)) \]
\[ = \partial v \wedge (-1)^{n-p} \partial \nu \]
which shows (3) in Proposition 3.1.

A \( p \)-tensor field \( v \) is called divergence-free, if \( \partial v = 0 \). (3) of Proposition 3.1 shows that the dual wedge product of two divergence-free tensor fields is divergence-free. But we can not expect this fact for the usual wedge product of \( p \)-tensor fields.[4]

Define the linear space of divergence-free \( p \)-tensors by
\[ Z_p(T, R) = \{ v \in T_p | \partial v = 0 \} \]
and the linear space of \( p \)-tensor field divergences by
\[ B_p(T, R) = \{ v \in T_p | v = \partial w \text{ for some } w \in T_{p+1} \} \].

We shall call \( H_p(T, R) \) by the de Rham homology group, where \( H_p(T, R) \) is a real coefficient group on \( M \) defined by \( Z_p(T, R)/B_p(T, R) \). (3.2) asserts that, for \( p \)-tensor field \( v \), \( \ast v \) is closed if and only if \( v \in Z_p(T, R) \), and \( \ast v \) is exact if and only if \( v \in B_p(T, R) \). Then the operator \( \ast \) induces an isomorphism between \( H_p(T, R) \) and the \((n-p)\)th de Rham cohomology group \( H^{n-p}(T^*, R) \) on \( M \). If \( M \) is compact, this result is nothing but the Poincaré duality theorem.

It is known that, if \( M \) is a connected orientable paracompact \( C^\infty \)-manifold, the \((n-p)\)th de Rham cohomology group \( H^{n-p}(T^*, R) \) on \( M \) is isomorphic onto the \( p \)-th real coefficient singular homology group \( H_p(S^M) \). On the other hand, it is known that if \( M \) is paracompact \( H_p(S^M) \) is isomorphic onto \( H_p(M, R) = \lim \limits_{\rightarrow} H_p(U, R) \), where \( H_p(U, R) \) is the real coefficient homology group of a covering \( U \) of \( M \) and \( \lim \) means the projective limit for coverings of \( M \). The group \( H_p(M, R) \) is called a \( p \)-th real coefficient homology group of \( M \).

**Proposition 3.2.** Let \( M \) be an orientable connected paracompact \( C^\infty \)-manifold of the dimension \( n \). The \( p \)-th de Rham homology group \( H_p(T, R) \) on \( M \) is isomorphic with the \( p \)-th real coefficient singular homology group \( H_p(S^M) \) on \( M \) and the \( p \)-th real coefficient homology group of \( M \), i.e., \( H_p(M, R) \).

**4. The \( D \)-affine connection.** In the previous section, we have defined the divergence of the \( p \)-tensor field. In this section, we shall consider the question about the extension of the divergence from on \( p \)-tensor fields to on the space \( T^*_Y \). For the purpose we shall define the \( D \)-affine connection which is considered as a dual of the usual affine connection in a certain sense. Let \( M \) be an orientable \( C^\infty \)-manifold.
The Formulation of D-affine Connection

**DEFINITION 4.1.** A D-affine connection on $M$ is an operator $\partial$ on $T^{n-1}_n$ into $T^{n-1}_{n-1}$ satisfying the following two properties:

1. $\partial(X + Y) = \partial X + \partial Y$,
2. $\partial(aX) = (\partial \! a)X + a\partial X$,

where $a \in \mathbb{R}$ and $d$ is the exterior differentiation operator.

We shall construct the unique operator on each $T^{0}_p$ to $T^{2}_{p-1}$ by virtue of the operator $\partial$ which defines the D-affine connection.

Now let us consider the mapping

$$\prod_{i=1}^{n-q} X_i \mapsto \sum_{j=1}^{n-q} \prod_{i=1}^{j-1} X_i \land \prod_{i=j+1}^{n-q} X_i \land \partial X_j,$$

where $X_1, \ldots, X_{n-q} \in T^{n-1}_n$, hence the mapping is on $(n-q)$-tuples of elements of $T^{0}_1$ into $T^{1}_{n-q}$. This is alternating and multilinear, hence there exists a linear mapping $f$ on $T^{0}_{n-q}$ to $T^{1}_{n-q}$ such that

$$f(\prod_{i=1}^{n-q} X_i) = \sum_{j=1}^{n-q} \prod_{i=1}^{j-1} X_i \land \prod_{i=j+1}^{n-q} X_i \land \partial X_j$$

for any elements $X_1, \ldots, X_{n-q}$ in $T^{n-1}_n$. By employing (2. 5), we have

$$\prod_{i=1}^{n-q} X_i \land \prod_{i=n-q}^{n-1} X_i = \prod_{i=1}^{n-q} (X_i \land \partial X_j),$$

and

$$\sum_{j=1}^{n-q} \prod_{i=1}^{j-1} X_i \land \prod_{i=n-q}^{n-1} X_i \land \partial X_j = \prod_{j=1}^{n-q} \prod_{i=1}^{n-q} (X_i \land \partial X_j).$$

Hence we have

$$\sum_{j=1}^{n-q} \prod_{i=1}^{j-1} X_i \land \prod_{i=n-q}^{n-1} X_i \land \partial X_j = \prod_{j=1}^{n-q} \prod_{i=1}^{n-q} (X_i \land \partial X_j).$$

Putting $\partial = \#^{-1}f\#$ on $T^{0}_n$, we have

$$\partial(X_1 \land \cdots \land X_{n-q}) = \sum_{j=1}^{n-q} X_1 \land \cdots \land X_{j-1} \land \partial X_j \land X_{j+1} \land \cdots \land X_{n-q},$$

where $X_1 \land \cdots \land X_{n-q} \in T^{0}_n$ and the right hand side of (4. 1) belongs to $T^{0}_{n-1}$. Then we have a linear mapping $\partial$ on $T^{0}_n$ into $T^{0}_{n-1}$.

Consider $X \in T^{0}_p$ by linearity, it suffices to consider the case $X = v \circ \omega$, where $v \in T_p$ and $\omega \in T^0$. Uniquely we can put

$$X = v \circ \omega = v^o X',$$

as a element of $T^0 p T^0_n$, where $X' = \partial \circ \omega$.

Consider the mapping

$$\prod_{i=1}^{n-q} (\# v, \# X') \mapsto \#(\partial v) \land \# X' + (-1)^{n-p} \# v \land \#(\partial X'),$$
where \( v \in T_p \) and \( X' \in T'_p \). This mapping is bilinear, hence there exists a linear mapping \( \tilde{f} \) on \( \#T_p \otimes \#T'_p \) into \( T^{n-1}_{p-1} \) such that

\[
\tilde{f}(\#v \wedge \#X') = \#(\tilde{f}v) \wedge \#X' + (-1)^{n-p} \#v \wedge \#(\tilde{f}X')
\]

for any \( v \in T_p \) and any \( X' \in T'_p \). By (2.5), we have

\[
\#v \wedge \#X' = \#(v_0 X')
\]

and

\[
\#(\tilde{f}v) \wedge \#X' + (-1)^{n-p} \#v \wedge \#(\tilde{f}X') = \#((\tilde{f}v_0 X') + (-1)^{n-p}(v_0 \tilde{f}X')).
\]

Hence we have

\[
\#f^{-1} \#(v_0 X') = \tilde{f}v_0 X' + (-1)^{n-p}(v_0 \tilde{f}X').
\]

Putting \( \tilde{f} = \#^{-1}f\# \) on \( T_p \), we can conclude

(4.3)

\[
\tilde{f}(v_0 X') = \tilde{f}v_0 X' + (-1)^{n-p}(v_0 \tilde{f}X'),
\]

where \( X = v_0 X' \in T_p \) and the right hand side of (4.3) belongs to \( T^{n-1}_{p-1} \). Thus there exists a linear mapping \( \tilde{f} \) on \( T_p \) into \( T^{n-1}_{p-1} \). We can easily verify the uniqueness of such \( \tilde{f} \).

**PROPOSITION 4.2.** If \( \tilde{f} \) is a \( D \)-affine connection on \( M \), then there exists a unique collection of operators, also denoted by \( \tilde{f} \), one on each space \( T_p \) to \( T^{n-1}_{p-1} \) \((q < n)\) and satisfying the following properties:

1. \( \tilde{f}(X + Y) = \tilde{f}X + \tilde{f}Y \) for \( X \in T_p \) and \( Y \in T_p \),
2. \( \tilde{f}(X \circ Y) = \tilde{f}X \circ Y + (-1)^{n-p} X_0 \circ Y \) for \( X \in T_p \), \( Y \in T_p \),
3. \( \tilde{f} \) coincides with the given \( D \)-affine connection on \( T^{n-1}_p \), and coincides with the divergence on \( T_p \).

For \( X \in T_p (p = 1, \ldots, n) \) we define

(4.4)

\[
\tilde{f}X = \#d\#X,
\]

where \( d \) is the exterior differentiation operator. Hence \( \tilde{f}X = 0 \), if \( X \in T_p \) \((p = 1, \ldots, n)\).

To prove the Proposition 4.2, the proof of (2) remains. By linearity it suffices to prove this in the following special case: \( X = v_0 X' \) and \( Y = w_0 Y' \), where we take the identification (4.2) and \( X \in T_p \), \( Y \in T'_p \) and \( v \in T_p \), \( w \in T'_p \), \( X' \in T'_p \), \( Y' \in T'_p \). Hence

\[
\tilde{f}(X \circ Y) = \tilde{f}(v_0 X' \circ w_0 Y')
\]

\[
= (-1)^{n(p-n)} \tilde{f}(v_0 w_0)(X' \circ Y')
\]

\[
= (-1)^{n(p-n)} \{\tilde{f}(v_0 w_0)(X' \circ Y') + (-1)^{n-p} (v_0 w_0) \circ \tilde{f}(X' \circ Y')\}
\]

\[
= (-1)^{n(p-n)} \{\tilde{f}(v_0 w_0) + (-1)^{n-p} v_0 \circ \tilde{f} w_0\} \circ (X' \circ Y')
\]

\[
+ (-1)^{(n-p)+(n-p)} (v_0 w_0) \circ (X' \circ Y')
\]

\[
= (-1)^{(n-m)+(n-p)} ((v_0 w_0 \circ (X' \circ Y') + (-1)^{n-p} (v_0 \circ \tilde{f} w_0) \circ (X' \circ Y'))
\]

\[
+ (-1)^{(m-p)} (v_0 w_0 \circ (X' \circ Y') + (v_0 w_0 \circ (X' \circ Y')))\}
The Formulation of $D$-affine Connection

\[
= (\partial_v X') \circ (w \circ Y') + (-1)^{n-p}(v \circ X') \circ (w \circ Y') \\
+ (-1)^{n-p-p'+(n-p') \circ (v \circ X') \circ (w \circ Y') + (-1)^{n-p-p'}(v \circ X') \circ (w \circ Y') \\
= (v \circ X') \circ (w \circ Y') + (-1)^{n-p}(v \circ X') \circ (w \circ Y') \\
= \partial X \circ Y + (-1)^{n-p}X \circ Y.
\]

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