ON AFFINE TRANSFORMATION IN A II-DECOMPOSABLE MANIFOLD OF THE TYPE (2, 2)

TAKUYA SAEKI

(2, 2) 型のII-分解可能な多様体の擬似変換について

Recently M. Obata [6] has proved that the group of all affine transformations of an almost complex manifold preserving the structure under some assumptions. On the other hand, we can consider on a differentiable manifold of the dimension $m$ (not necessarily even) the complex almost-product structure or II-structure introduced by D. C. Spencer and it is one of the generalizations of the almost complex structure.

Then the author has proved in [7] that the connected component of the identity of the group of all affine transformations preserves the II-structure if the manifold is II-irreducible and if $\dim T_1 = \dim T_2$ where $T_1$ and $T_2$ are distributions defining the II-structure. But this result is for the II-decomposable case of the type (1, 1). In the present paper, we shall study the same question for the II-decomposable case of the type (2, 2).

In §1, we shall make a brief sketch of the II-structure and the II-connection. In §2, we shall define the II-decomposability. In §3, we shall prove the main theorem.

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1. Let $M$ be an $m$-dimensional differentiable manifold of class $C^\infty$. We denote by $T_x$ the tangent vector space of $M$ at $x \in M$ and by $T^c_x$ its complexification.

A II-structure (or complex almost-product structure in the terminology of D. C. Spencer) is defined on $M$ by giving two fields $T_1$ and $T_2$, of class $C^\infty$, of complementary proper subspaces of $T^c_x$. If we set $n_1 = \dim T_1$ and $n_2 = \dim T_2$, then we have $n_1 + n_2 = m$, $n_1 \neq 0$, $n_2 \neq 0$.

Let $P_1$ (resp. $P_2$) be the projection of $T^c_x$ onto $T_1$ (resp. $T_2$) at every point $x \in M$. All vectors $v$ of $T^c_x$ is the sum of a vector $P_1v \in T_1$ and of a vector $P_2v \in T_2$. Let $\lambda$ be a complex constant which is not zero. If we set

\[ (1.1) \quad Iv = \lambda(P_1v - P_2v), \]

then we can define a linear operator $I$ on complex vector space $T^c_x$ such that

\[ (1.2) \quad I^2 = \lambda^2 \text{ (Identity)}. \]

To the operator $I$ corresponds a complex tensor $(F_f)$ defined by $(Io)^f = F_f^i v^j$. We can characterize the II-structure by the above tensor field $(F_f)$ which satisfies

\[ (1.2) \quad F_{ah}^i F_h^j = \lambda^2 \delta^i_j. \]

A base of $T^c_x$ is called a complex base relative to $x$. The set $E^c(M)$ of complex

1) Numbers in brackets refer to the bibliography at the end of the paper.
bases relative to different points of $M$ admits a structure of a principal fibre bundle whose base space is $M$ and whose structure group is $GL(m, \mathbb{C})$. An infinitesimal connection in $E^c(F)$ is called a complex linear connection.

A base $(e_\alpha)$ of $T_x^c$ such that $e_\alpha \in T_x$, $e_\beta \in T_x$ ($\alpha = 1, \ldots , n_1, \beta = n_1 + 1, \ldots , m$) is called a $\Pi$-adapted base relative to $x$. The set $E_\Pi(M)$ of $\Pi$-adapted bases relative to different points of $M$ admits a structure of a principal fibre bundle whose base space is $M$ and whose structure group is a subgroup $G(n_1, n_2)$ of $GL(m, \mathbb{C})$ constructed by the matrices

$$
\begin{pmatrix}
A & 0 \\
0 & B
\end{pmatrix}
$$

where $A \in GL(n_1, \mathbb{C})$ and $B \in GL(n_2, \mathbb{C})$. $G(n_1, n_2)$ is isomorphic to $GL(n_1, \mathbb{C}) \times GL(n_2, \mathbb{C})$. An infinitesimal connection in $E_\Pi(M)$ is called a $\Pi$-connection. It is known that a complex linear connection can be identified to a $\Pi$-connection if and only if the tensor $(F_i^j)$ is covariant constant.

A transformation $\varphi$ of $M$ is called an affine transformation for the (real) linear connection if it preserves the connection. Now we shall define the affine transformation for the complex linear connection. For the purpose, we consider a real $\Pi$-structure (we denote briefly: $\Pi$-structure) on $M$, which is defined by giving two fields $T_1$ and $T_2$, of class $C^\infty$, of complementary proper subspaces of $T_x$. The set $E_\Pi^R(M)$ of $\Pi$-adapted bases relative to different points of $M$ admits a structure of a principal fibre bundle. An infinitesimal connection in $E_\Pi^R(M)$ is called a $\Pi$-connection. All $\Pi$-connection $\omega$ of $E_\Pi^R(M)$ defines a $\Pi$-connection of $E_\Pi(M)$ with which the $\Pi$-connection can be identified. Hence for a $\Pi$-connection $\omega$ and an affine transformation $\varphi$, we have $\varphi \omega = \omega$.

$A(M)$ denotes the Lie group of all affine transformations of $M$ onto itself and $A_0(M)$ denotes the connected component of the identity of $A(M)$.

2. Let $M$ be an $m$-dimensional differentiable manifold with the $\Pi$-structure. Hereafter we shall assume that $M$ satisfies the second countability axiom, then the principal fibre bundle over $M$ has always an infinitesimal connection. Let $H(x)$ be the holonomy group of the $\Pi$-connection at a point $x \in M$, it is a Lie subgroup of $G(n_1, n_2)$ and relative to the $\Pi$-adapted base the form of its element becomes

$$
\begin{pmatrix}
A & 0 \\
0 & B
\end{pmatrix},
$$

where $A$ acts on $T_1$ and $B$ on $T_2$.

Now we shall define the $\Pi$-decomposability of the type $(s, t)$. $M$ is called to be $\Pi$-decomposable of the type $(s, t)$ if the following condition is fulfilled:

$H(x)$ is $D_1$-irreducible in $C$, i.e., $T_1 = T_1^{(1)} + T_1^{(2)} + \cdots + T_1^{(t)}$, $T_2 = T_2^{(1)} + T_2^{(2)} + \cdots + T_2^{(s)}$, $T_1^{(k)}(k = 1, \ldots , t)$, $T_2^{(l)}(l = 1, \ldots , s)$ are all invariant by $H(x)$, and $H(x)$ is irreducible in each $T_1^{(k)}(k = 1, \ldots , t)$ and $(T_2^{(l)}(l = 1, \ldots , s))$.

This notion is invariant of the choice of $x$. A base $(e_\alpha)$ of $T_x^c$ such that $e_\alpha \in T_1^{(i)}(\alpha^i = 1, \ldots , \dim. T_1^{(i)})$, $e_\beta \in T_2^{(j)}(\alpha^j = \dim. T_1^{(i)} + \cdots + \dim. T_1^{(i-1)} + 1, \ldots , n_1; \beta = \dim. T_1^{(i)} + \cdots + \dim. T_2^{(j)})$, $e_\mu \in T_2^{(j)}(\mu = n_1 + 1, \ldots , n_2 + \dim. T_2^{(j)}, \ldots , e_{\mu_0} \in T_2^{(j)}(\mu^0 = n_1 + \dim. T_2^{(j)} + \cdots + \dim. T_2^{(i-1)} + \cdots , m; \mu = \dim. T_2^{(j)} + \cdots + \dim. T_2^{(i-1)} + \cdots , m)$.
On Affine Transformation in a $\pi$-Decomposable Manifold of the Type (2,2)
THEOREM. Let $M$ be an $m$-dimensional differentiable manifold with the $\Pi$-structure. Assume that $M$ is $\Pi$-decomposable of the type $(2,2)$ and that every $p_i (i = 1, 2, 3, 0)$ differs from each other. Then $M$ has three independent $\Pi$-structures $E_1$, $E_2$ and $E_3$ such that

$$E_1 E_2 = F_2 E_3 = - \lambda E_4,$$

$$E_2 E_3 = E_3 F_1 = - \lambda E_1,$$

$$E_3 E_1 = E_1 E_2 = - \lambda E_4$$

and they are all parallel and $A_0 (M)$ preserves these $\Pi$-structures.

PROOF. Let $F$ be the tensor field defining the $\Pi$-structure on $M$. For $x \in M$, we shall denote $F_x$ the value of the tensor $F$ at $x$. Relative to the $G$-adapted base, the tensor $F_x$ has one form of three forms (3.2). Since $F_x$ is invariant under the holonomy group $H(x)$ of the $\Pi$-connection. The matrix $F_x$ commutes with any element of $H(x)$, so that $F_x$ is a commutator of $H(x)$.

In general, let $\mathfrak{R}$ be a commutator algebra of $H(x)$ and $K \in \mathfrak{R}$ with

$$K = \begin{pmatrix} K_1 & K_2 & H_1 & H_2 \\ K_3 & K_4 & H_3 & H_4 \\ L_1 & L_2 & M_1 & M_2 \\ L_3 & L_4 & M_3 & M_4 \end{pmatrix}$$

where $K_1$, $K_4$, $M_1$ and $M_4$ are square matrices and others are all rectangular matrices. For an element of $H(x)$ of the form (3.1) relative to the $D_2^2$-adapted base, we have

$$K_1 A_1 = A_1 K_1, \quad K_2 A_2 = A_2 K_2, \quad H_1 B_1 = A_1 H_1, \quad H_2 B_2 = A_2 H_2,$$

$$K_3 A_1 = A_3 K_3, \quad K_4 A_2 = A_4 K_4, \quad H_3 B_1 = A_2 H_3, \quad H_4 B_2 = A_4 H_4,$$

$$L_1 A_1 = B_1 L_1, \quad L_2 A_2 = B_2 L_2, \quad M_1 B_1 = B_1 M_1, \quad M_2 B_2 = B_1 M_2,$$

$$L_3 A_1 = B_2 L_3, \quad L_4 A_2 = B_3 L_4, \quad M_3 B_1 = B_2 M_3, \quad M_4 B_2 = B_3 M_4.$$ And these hold identically in $h$ of $H(x)$.

Let $A_i (h)$ (resp. $B_i (h)$) $(i = 1, 2)$ be a matrix of $A_i$ (resp. $B_i$) $(i = 1, 2)$ of an element of $H(x)$ of the form (3.1). Let $H_i' (resp. H_i'')$ $(i = 1, 2)$ be a set of $A_i (h)$ (resp. $B_i (h)$) $(i = 1, 2)$ for all $h$ of $H(x)$. Since $M$ is $\Pi$-decomposable, $H_i$ and $H_i'' (i = 1, 2)$ are irreducible in $C$. On the other hand, for each element $A_i (h)$ (resp. $B_i (h)$) $(i = 1, 2)$ of $H_i'' (resp. H_i'')$ $(i = 1, 2)$ we have $K_1 A_i (h) = A_i (h) K_1, \quad K_2 A_i (h) = A_i (h) K_2, \quad K_3 A_i (h) = A_i (h) K_3, \quad K_4 A_i (h) = A_i (h) K_4, \quad M_1 B_i (h) = B_i (h) M_1, \quad M_2 B_i (h) = B_i (h) M_2, \quad M_3 B_i (h) = B_i (h) M_3, \quad M_4 B_i (h) = B_i (h) M_4$ for all $h$ of $H(x)$. By Lemma 2, we have

$$K_1 = \alpha_1 I_{p_1}, \quad K_4 = \alpha_4 I_{p_3}, \quad M_1 = \beta_1 I_{p_1}, \quad M_4 = \beta_4 I_{p_3}, \quad \alpha_1, \quad \alpha_2, \quad \beta_1, \quad \beta_2 \in C.$$ On the other hand, for all $h$ of $H(x)$ we have

$$K_3 A_i (h) = A_2 (h) K_3, \quad K_4 A_i (h) = A_1 (h) K_4, \quad H_1 B_i (h) = A_1 (h) H_1, \quad H_2 B_i (h) = A_1 (h) H_2,$$

$$L_1 A_i (h) = B_1 (h) L_1, \quad L_2 A_i (h) = B_2 (h) L_2, \quad H_3 B_i (h) = A_2 (h) H_3, \quad H_4 B_i (h) = A_2 (h) H_4,$$

$$L_3 A_i (h) = B_2 (h) L_3, \quad L_4 A_i (h) = B_3 (h) L_4, \quad M_3 B_i (h) = B_2 (h) M_3, \quad M_4 B_i (h) = B_3 (h) M_4$$

identically. By Lemma 3, we have

3) The equations (3.3) shows that these $\Pi$-structures form a certain extension of some structure which is similar to the quaternion structure of the type $\Pi$ and $\Pi$ in C.J.Hsu [1].
On Affine Transformation in a $\mathfrak{w}$-Decomposable Manifold of the Type (2.2)

\begin{align*}
K_1 &= 0, & K_2 &= 0, & H_1 &= 0, & H_2 &= 0, \\
L_1 &= 0, & L_2 &= 0, & H_3 &= 0, & H_4 &= 0, \\
L_3 &= 0, & L_4 &= 0, & M_1 &= 0, & M_2 &= 0.
\end{align*}

Then we can conclude that an arbitrary element $K$ of the commutator algebra $\mathfrak{K}$ of $H(x)$ has the form

$$K = \begin{pmatrix}
\alpha_1 I_{p_1} & 0 & 0 & 0 \\
0 & \alpha_2 I_{p_2} & 0 & 0 \\
0 & 0 & \beta_1 I_{p_3} & 0 \\
0 & 0 & 0 & \beta_2 I_{p_4}
\end{pmatrix}.$$ 

Thus $\mathfrak{K}$ is spanned by $E_0$, $E_1$, $E_2$ and $E_3$ of (3.2) with the relations (3.3), where

$$E_0 = \begin{pmatrix}
\lambda I_{p_1} & 0 & 0 & 0 \\
0 & \lambda I_{p_2} & 0 & 0 \\
0 & 0 & \lambda I_{p_3} & 0 \\
0 & 0 & 0 & \lambda I_{p_4}
\end{pmatrix}.$$ 

Let $P^c(1,1)$ be the vector space spanned by all parallel tensor fields of the type (1, 1) on $M$ and $\widetilde{P}^c(1,1)$ the subset of all the element $K$ of $P^c(1,1)$ such that $K^2 = \lambda^2 I_m$, i.e., $K$ is a II-structure of $M$. Then any element $\varphi$ of $A(M)$ transforms linearly $\widetilde{P}^c(1,1)$ onto itself. Indeed, since the tensor field $I_m$ is invariant under all transformations we have

$$(\varphi(\varphi) K)^2 = \lambda^2 \varphi(\varphi) I_m$$

for every $K \in \widetilde{P}^c(1,1)$ where $\varphi$ is a homomorphism of $A(M)$ into $GL(p, C)$ defined by $\varphi(\varphi)t = \varphi t$ for any $t \in P^c(1,1)$ any $t \in P^c(1,1)$ and $\varphi = \dim. P^c(1,1)$. Assigning $K \in P^c(1,1)$ to the value $K_x$ of $K$ at $x \in M$, $P^c(1,1)$ is isomorphic with the subspace of the tensor space of the type (1, 1) over $T_x$ consisting of all tensors invariant under $H(x)$, i.e., $P^c(1,1)$ is isomorphic with the commutator algebra $\mathfrak{R}$ of $H(x)$. It is obvious that $\widetilde{P}^c(1,1)$ is isomorphic with the subset $\widetilde{\mathfrak{K}}$ of $\mathfrak{K}$ consisting of the commutators $K$ such that $K^2 = \lambda^2 I_m$.

An arbitrary element $K$ of $\mathfrak{K}$ is described by

$$K = a_0 E_0 + a_1 E_1 + a_2 E_2 + a_3 E_3, \quad a_i \in C(i = 0, 1, 2, 3).$$

If $K \in \widetilde{\mathfrak{K}}$, we have

$$a_0 \alpha_1 = a_2 \alpha_2, \quad a_0 \alpha_3 = a_1 \alpha_1, \quad a_0 \alpha_3 = a_1 \alpha_3, \quad \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1.$$ 

Solving these equations, we can determine the elements of $\widetilde{\mathfrak{K}}$, i.e., $\widetilde{\mathfrak{K}}$ consists of the elements $\pm E_0, \pm E_i, \pm E_3$ and $1/2(\varepsilon_0 E_0 + \varepsilon_i E_i + \varepsilon_2 E_2 + \varepsilon_3 E_3)$ where $\varepsilon_i = \pm 1$ $(i = 0, 1, 2, 3)$. But we abandon the case when the tensor in proportion to $I_m$, hence we have a II-structure by displacing $E_i(i = 1, 2, 3)$ parallelly. Since $\rho(\varphi) \widetilde{P}^c(1,1) \subset \widetilde{P}^c(1,1)$ for every $\varphi \in A(M)$, we have $\rho(\varphi)E_i = \pm E_i$ or $\pm E_i(i, j = 1, 2, 3; i \neq j)$. As $\rho$ is continuous, we have $\rho(\varphi)E_i(i = 1, 2, 3)$ for every $\varphi \in A_0(M)$, i.e., $A_0(M)$ preserves the II-structure $E(i = 1, 2, 3)$. q.e.d.
Bibliography

[1] C. J. Hsu, On some structures which are similar to the quaternion structure, to appear.

IWATE UNIVERSITY,
MORIOKA, JAPAN.