SEMIGROUPS IN WHICH ANY PARTITION IS DECOMPOSITION

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任意の分割が分解となる単群について
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When a set $M$ is divided into the class-sum of some disjoint subsets:
$$
M = \bigcup_i M_i, \quad M_i \cap M_j = \emptyset \quad (i \neq j),
$$
this is called a partition of the set $M$. When a semigroup $S$ is divided into the
class-sum of some disjoint subsets such that
$$
S = \bigcup_{x \in T} S_x, \quad S_x \cap S_y = \emptyset \quad (x \neq y),
$$
and for $x, y \in T$, there is $z \in T: S_x S_y \subseteq S_z$, then this is called a decomposition of
the semigroup $S$.

Now we should like to present the following problem: What is a semigroup $S$
eq
every partition of which is a decomposition?

Such a semigroup shall be called a $\mathfrak{p}$-semigroup. The purpose of this short note
is to determine the structure of $\mathfrak{p}$-semigroups. The semigroups of order $\leq 2$ are the
trivial cases of $\mathfrak{p}$-semigroups. Hereafter let $S$ be an infinite or finite $\mathfrak{p}$-semigroup of
order $\geq 3$.

We shall proceed to discuss about the two cases: the first in which $S$ has at
least a non-idempotent, the second in which $S$ is idempotent. We shall use the
notation $\Delta(a_1, \ldots, a_k)$ which represents the partition of $S$ which is divided into two
classes:
$$
S = \{a_1, \ldots, a_k\} \cup (S - \{a_1, \ldots, a_k\}).
$$

1. In the case where $S$ has a non-idempotent $a$.

Let $b = a^2 \neq a$. Consider the partition $\Delta(b)$ of $S$. Since $\Delta(b)$ is a decomposition
of $S$, referring $a^2 = b$ and $a \neq b$, we have
$$
xy = b \quad \text{for any } x \neq b, \ y \neq b;
$$
especially
$$
x^2 = b \quad \text{for all } x \neq b.
$$

Take any two distinct elements $u$ and $v$ which are both different from $b$. This
is possible as the order has been assumed to be $\geq 3$.

Making two decompositions $\Delta(u, v)$ and $\Delta(v, u)$, from $u^2 = b$ and $v^2 = b$ because of
(2), it follows that $b^2 \in \{b, u\} \cap \{b, v\}$
and hence we get
$$
b^2 = b.
$$

*) $\{a_1, \ldots, a_k\}$ denotes the set which consists of the elements $a_1, \ldots, a_k$. 

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By (1) we have
\[ u(S - \{b, u\}) = (S - \{b, u\})u = v(S - \{b, v\}) = (S - \{b, v\})v = \{b\}. \]

Utilizing this formula we obtain easily
\[
(4) \quad b(S - \{b, u\}) \subseteq \{b, u\}, \quad (4') \quad (S - \{b, u\})b \subseteq \{b, u\}, \\
(5) \quad b(S - \{b, v\}) \subseteq \{b, v\}, \quad (5') \quad (S - \{b, v\})b \subseteq \{b, v\}.
\]

Immediately we have
\[
(6) \quad bx = b \quad \text{for all } x \in S - \{b, u, v\}.
\]

Further by (4) and (5) with \( u \in S - \{b, v\}, \ v \in S - \{b, u\} \), we arrive at
\[
(7) \quad bu = b \text{ or } v, \\
(8) \quad bv = b \text{ or } u.
\]

On the other hand, suppose
\[
(9) \quad bu = v,
\]
then by (3) we have
\[
(10) \quad bu = bv = b \text{ or } u, \text{ contradicting (9)}.
\]

Therefore
\[
(11) \quad bu = b.
\]

Similarly
\[
(12) \quad bv = b.
\]

By (3), (6), (11), (12), \( b \) must be a left zero i.e.
\[
(13) \quad bx = b \quad \text{for all } x \in S.
\]

Similarly, from (4'), (5'), we get
\[
(14) \quad xb = b \quad \text{for all } x \in S.
\]

Gathering (1), (13) and (14) together, it is concluded that if \( S \) has a non-idempotent \( a \), then \( S \) is a semigroup defined as
\[
xy = a^2 \quad \text{for all } x, y \in S.
\]

2. In the case where \( S \) is idempotent.

Let \( a, b \) be any two distinct elements of \( S \) and make a decomposition \( \Delta_{(a,b)} \).

Since \( a^2 = a \), \( ab = a \) or \( b \). If \( ab = a \), then
\[
(15) \quad xb = b \quad \text{for } x \neq b
\]
because of \( \Delta_{(a)} \); and using \( b^2 = b \) and \( \Delta_{(a,b)} \) where \( x \neq a, \ x \neq b \), it is derived that
\[
x = b \text{ or } x, \text{ and by (15)}
\]
\[
x = b \quad \text{for } x \neq a, \ x \neq b.
\]

Since \( ab = a \) and \( b^2 = b \), we get
\[
xb = x \quad \text{for all } x \in S.
\]
Such a semigroup $S$ is called left singular. If $ab = b$, then we have
\[ ax = x \quad \text{for all } x \in S \]
in the similar way. Such an $S$ is called right singular.

Conversely we see easily that the semigroups of three types, a zero-semigroup defined as $xy = 0$ (for all $x, y$), a right singular semigroup, and a left singular semigroup are $\mathfrak{p}$-semigroups.

Thus we have

**Theorem.** A non-trivial $\mathfrak{p}$-semigroup is one of the following semigroups.
1. a zero-semigroup defined as $xy = 0$ for all $x, y$.
2. a right singular semigroup.
3. a left singular semigroup.

From the above theorem we have immediately that a subsemigroup of a $\mathfrak{p}$-semigroup is a $\mathfrak{p}$-semigroup and the homomorphic image of a $\mathfrak{p}$-semigroup is also a $\mathfrak{p}$-semigroup.

As we have stated, we used only the partitions of type
\begin{equation}
S = A \cup B \text{ where } A \text{ consists of one or two elements.}
\end{equation}

Accordingly $\mathfrak{p}$-semigroup is characterized by a weakened condition:

A semigroup whose all partitions of type (16) are decompositions.

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