Y. Muto\(^1\) investigated some properties of a fibred Riemannian manifold and proved a theorem concerning the local decomposition of the manifold. And S. Ishihara \(^3\) \(^4\) investigated the same problem globally. In this paper we shall consider the decomposition problem in a fibred Hermitian space.

Let us suppose that indices run as follows:
\[
\begin{align*}
  a, d, c, & = 1, \ldots, n, \\
  i, j, k, & = n+1, \ldots, n+m, \\
  \alpha, \beta, \gamma, & = 1, \ldots, n, n+1, \ldots, n+m, \\
  A, B, C, & = 1, \ldots, n, n+1, \ldots, n+m. 
\end{align*}
\]

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§ 1. Fibred Hermitian space. In this section we shall give the definition of the fibred Hermitian space and its fundamental properties.

Let \(B(X, Y)\) be an analytic fibre bundle\(^2\) with the \(2n\)-dimensional base space \(X\) and with the \(2m\)-dimensional fibre \(Y\). The topological dimension of the bundle space is \(2(n+m)\). And we shall assume that it has an usual \((n+m)\)-dimensional complex analytic structure. Moreover, we shall consider in the manifold \(B\) a system of favourable coordinate neighbourhoods introduced in § 1 of \(C\). But this time, the transformation between two favourable coordinate systems are expressed by the equations
\[
\begin{align*}
  z^a' &= z^a'(z^1, \ldots, z^n); \text{ conj.}, \\
  z^i' &= z^i'(z^1, \ldots, z^n; z^{n+1}, \ldots, z^{n+m}); \text{ conj.},
\end{align*}
\]
where \((z^a, \bar{z}^a)\) and \((z^a', \bar{z}^a')\) are two favourable coordinate systems of a point of \(B\) and the sign "conj." denotes the complex conjugates of the formulas already written.

Let us consider on an analytic complex manifold \(B\) a positive definite, symmetric and self-adjoint Hermitian metric tensor whose components are \(g_{ab}\) such that
\[
g_{ij} = g_{ia} = g_{ai} = 0; \text{ conj.}
\]
Hence the complex analytic manifold \(B\) becomes an analytic Hermitian space with the metric

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\(^1\) Numbers in brackets refer to the bibliography at the end of the paper.
\(^2\) See p. 31 of C. Ehresmann [2].
An \((n+m)\)-dimensional analytic Hermitian space \(B\) is called a \textit{fibred Hermitian space} if its underlying manifold has a bundle structure \(B(X, Y)\), where the base space \(X\) and the fibre \(Y\) are supposed to be manifolds of complex dimensions \(n\) and \(m\) respectively.

There exists field \(F\) of \(2n\)-dimensional (topological dimension) plane-elements which are Hermitianly orthogonal to the tangent space of the fibre at each point of \(B\). The field \(F\) is given, under the similar calculation in a fibred Riemannian space \((5)\), by the equations

\[
dz^i + \Gamma^i_{a} dz^a = 0 \ ; \ \text{conj.,}
\]

where \(\Gamma^i_{a}\)'s (conj.) are analytic functions of \(z^a\) and \(z^a\) defined by

\[
\Gamma^i_{a} g_{ij} = g_{ja} \ ; \ \text{conj.}
\]

It is clear that the functions \(\Gamma^i_{a}\) are self-adjoint. Hereafter we shall assume that the field \(F\) defined \((3)\) is always with lifts like in §1 of \((7)\).

The components \(g_{ab}\) of a Hermitian metric tensor are transformed for the coordinate transformation \((1)\) by the following equations:

\[
g_{\alpha' \beta'} = \frac{\partial z^\alpha}{\partial z'^\alpha} \frac{\partial z^\beta}{\partial z'^\beta} g_{\alpha \beta}
\]

\[
g_{\alpha' \beta'} = \frac{\partial z^\alpha}{\partial z'^\alpha} \frac{\partial z^\beta}{\partial z'^\beta} + \frac{\partial z^\alpha}{\partial z'^\beta} \frac{\partial z^\beta}{\partial z'^\alpha} g_{\alpha \beta} \ ; \ \text{conj.}
\]

Hence the quantities \(\Gamma^i_{a}\) (conj.) are transformed by the following equations:

\[
\Gamma^i_{a'} = \Gamma^i_{a} \frac{\partial z'^a}{\partial z^a} + \frac{\partial z^a}{\partial z'^a} \frac{\partial z'^a}{\partial z^a} ; \ \text{conj.}
\]

There is a remarkable fact that the quantities \(\Gamma^i_{a' b'}\) \((\text{conj.})\) are transformed like components of a tensor in the bundle space \(B\) for the coordinate transformations \((1)\), i.e.,

\[
\Gamma^i_{a' b'} = \Gamma^i_{a' b'} \ ; \ \text{conj.}
\]

Hence

\[
\Gamma^i_{a b} = 0 \ ; \ \text{conj.}
\]

is invariant under a coordinate transformation \((1)\) in a fibred Hermitian space \(B\) with a field \(F\) with lifts. The condition \((6)\) is called \textit{A\_1-condition}.

The quantities \(\Gamma^i_{a b}\) (conj.) are transformed under \((1)\) by the following equations:

\[
\Gamma^i_{a' b'} = \frac{\partial z'^a}{\partial z^a} \frac{\partial z'^b}{\partial z^b} + \Gamma^i_{a b} \frac{\partial z'^a}{\partial z^a} \frac{\partial z'^b}{\partial z^b} + \Gamma^i_{a' c'} \frac{\partial z'^a}{\partial z^a} \frac{\partial z'^c}{\partial z^c} \ ; \ \text{conj.}
\]

If a fibred Hermitian space \(B\) with a field \(F\) with lifts satisfies \(A_1\)-condition, the quantities \(\Gamma^i_{a b}\) (conj.) are transformed like components of a tensor for a coordinate transformations \((1)\). Hence

\[
\Gamma^i_{a b} = 0 \ ; \ \text{conj.}
\]
is invariant under a coordinate transformation (1) in a fibred Hermitian space with a field F with lifts satisfying A,,-condition. The condition (7) is called A,,-condition. If a fibred Hermitian space B satisfies A,-, and A,-condition, the space B is called to satisfy A,-condition.

If a fibred Hermitian space B with a field F with lifts satisfies A,-condition, the quantities $\Gamma^{i}_{a}$ (resp. $\Gamma^{i}_{a}$) are functions of only $z$ and $z$ (resp. $\bar{z}$ and $\bar{z}$) do not contain $\bar{z}$ and $\bar{z}$ (resp. $z$ and $z$). Hence the theorem in the footnote in p. 4 of Mut85 is modified as follows:

**LEMMA.** Let B be a fibred Hermitian space with a field F with lifts satisfying A,-condition. If the field F is completely integrable, then we can choose a set of favourable coordinate system such that $\Gamma^{i}_{a}$ (conj.) and $g_{i}$ (conj.) vanish identically.

We can prove the following proposition as a corollary of Proposition 1 in § 1 of (7) and the above Lemma.

**PROPOSITION 1.** A field F with lifts of a fibred Hermitian space B satisfying A,-condition is completely integrable if and only if the quantities $\Gamma^{i}_{a}$ (conj.) vanish in all set of favourable coordinate systems in B.

§ 2. Isometric fibres. We can consider some special properties concerning the correspondence between consecutive fibres in a fibred Hermitian space $B(X, Y)$ with a field F with lifts satisfying A,-condition.

Fibres in $B(X, Y)$ are called to be isometric if the correspondence between $Y_{x}$ and $Y_{x+dx}$ as given by (3) is Hermitianly isometric, where $x$ and $x+dx$ are consecutive points of the base space X. Hence for isometric fibres we have the following conditions by the same way as in a fibred Riemannian space

$$F_{ij}=0, \quad F_{i\bar{j}}=0$$

where we put

$$F_{ij}=g_{ij}-g_{ij}k^{i}_{a}k^{j}_{a}, \quad F_{i\bar{j}}=g_{ij}-g_{ij}\Gamma_{a}^{i}k_{a}^{j}$$

If we put

$$\delta z^{i}=dz^{i}+\Gamma^{i}_{a}dz^{a}; \text{ conj.},$$

we have

$$ds^{2}=g_{ij}dz^{i}d\bar{z}^{j}+g_{\alpha\bar{\beta}}dz^{\alpha}d\bar{z}^{\beta}+g_{ij}dz^{i}d\bar{z}^{j}+g_{\alpha\bar{\beta}}dz^{\alpha}d\bar{z}^{\beta}.$$  

If the we define a new quantity by

$$G_{\alpha\bar{\beta}}=g_{\alpha\bar{\beta}}-\frac{1}{2}\Gamma_{a}^{i}\Gamma_{b}^{j}g_{ij},$$

then we have

$$ds^{2}=g_{ij}\delta z^{i}\delta \bar{z}^{j}+G_{\alpha\bar{\beta}}dz^{\alpha}d\bar{z}^{\beta}.$$  

Since $\det |g_{ij}| \neq 0$ we get $\det |G_{\alpha\bar{\beta}}| \neq 0$. The quantities $G_{\alpha\bar{\beta}}$ are transformed by the following equations:

$$G_{\alpha\bar{\beta}}^{\prime} \neq \frac{\partial x^{\alpha}}{\partial \bar{x}^{\alpha}} \frac{\partial \bar{z}^{\beta}}{\partial \bar{x}^{\beta}} G_{\alpha\bar{\beta}},$$

for a coordinate transformation (1).
Now consider a pair of corresponding points \((z^a, z^i, Z^J)\) and \((z^a + dz^a, z^i + dż^i, Z^J + \Gamma^J_a dz^a, Z^J + \Gamma^J_i dż^i)\) on \(Y_z\) and \(Y_{z+dz}\) respectively. The distance between them measured in \(B\) is given by
\[
dl = \sqrt{G_{ab}} \, dz^a \, dż^i
\]
as we can see easily from (9). If this distance depends only upon the pair of fibres, \(Y_z\) and \(Y_{z+dz}\), and not upon individual points, we say that the fibres are parallel. They are characterized by the equations
\[
G_{a\delta,i} = 0, \quad G_{ab,i} = 0.
\]
Lemma 1. In a fibred Hermitian space \(B(X,Y)\) with a field \(F\) with lifts satisfying \(A_1\)-condition, the Christoffel symbols \(\{a_{jk}\}\) vanish if and only if the quantities \(G_{a\delta,i}\) vanish.

Proof. If \(G_{a\delta,i} = 0\), then
\[
\{a_{jk}\} = -\frac{1}{2} \left( g^{aj} g_{ki} + g^{aj} g_{ki} + g^{aj} g_{bi} + g^{aj} g_{bi} \right) = 0.
\]
For,
\[
g^{aj} g_{kj} + g^{aj} g_{kj} = \delta^{\delta}_{j} = 0, \quad \text{conj.},
\]
hence
\[
g^{aj} g_{kj} + \Gamma^j_b g^{ba} = 0, \quad \text{conj.}.
\]
Since \(\det | g^{aj} | \neq 0\),
\[
g^{aj} + \Gamma^j_{ai} g^{ba} = 0, \quad \text{conj.}.
\]
We can conclude that
\[
\{a_{jk}\} = \frac{1}{2} \left( g^{aj} G_{b\delta,j} \right).
\]
On the other hand, by virtue of \(\{a_{jk}\}\) we have
\[
(g^{aj} + \Gamma^j_b g^{ba}) g_{b\delta} = 0.
\]
Since
\[
g^{aj} g_{b\delta} + g^{aj} g_{b\delta} = \delta^{b\delta}_{j}
\]
we can get
\[
\delta^{b\delta}_{j} - g^{aj} g_{b\delta} + \Gamma^j_b g^{ba} g_{b\delta} = 0
\]
or
\[
g^{aj} (g_{b\delta} - \Gamma^j_b g_{b\delta}) = \delta^{b\delta}_{j}.
\]
Hence we have
\[
\det | g^{aj} | \neq 0.
\]
It is easy to see the lemma form \(\{a_{jk}\}\) and \(\{a_{jk}\}\).

Lemma 2. In a fibred Hermitian space \(B(X,Y)\) with a field with lifts satisfying \(A_1\)-condition, the Christoffel symbols \(\{a_{jk}\}\) vanish.

Proof. Employing \(\{a_{jk}\}\), we have
\[
\{a_{jk}\} = -\frac{1}{2} \left( g^{aj} g_{b\delta,j} + g^{aj} g_{b\delta,j} + g^{aj} g_{b\delta,j} + g^{aj} g_{b\delta,j} \right) = 0.
\]
Lemma 3. In a fibred Hermitian space \( B(X,Y) \) with a field \( F \) with lifts satisfying \( A_1 \)-condition, if fibres in \( B(X,Y) \) are isometric, then the Christoffel symbols \( \{ \alpha_j \} \) vanish.

Proof. Similarly as Lemma 2,
\[
\{ \alpha_j \} = \frac{1}{2} \left( g^{ai} g_{bi,j} - g^{ai} g_{bj,i} + g^{ab} g_{bi,c} - g_{bi,c} \right)
\]
\[
= \left( g^{ia} + \Gamma^a_{jk} g^{ab} \right) \left( g_{bi,j} - g_{bj,i} \right) - g^{ab} F_{j,k}
\]
\[
= 0.
\]

Lemma 4. In a fibred Hermitian space \( B(X,Y) \) with a field \( F \) with lifts satisfying \( A \)-condition whose fibres are isometric, the Christoffel symbols \( \{ \alpha_j \} \) (conj.) vanish if and only if the quantities \( R_{\alpha,\beta} \) (conj.) vanish.

Proof. \[
\{ \alpha_j \} = \frac{1}{2} \left( g^{ab} g_{\alpha,b} - g^{ab} g_{\beta,a} + g^{ac} g_{\alpha,c} - g_{\alpha,c} \right)
\]
\[
= \frac{1}{2} g^{ab} \left( -g_{\alpha,b} \Gamma^b_{\alpha} + g_{\beta,a} \Gamma^a_{\alpha} + g_{\alpha,c} - g_{\alpha,c} \right)
\]
\[
= \frac{1}{2} g^{ab} \left( g_{\alpha,b} \bar{R}_{\alpha,\beta} - F_{j,k} \Gamma_{\alpha}^j \right)
\]
\[
= \frac{1}{2} g^{ab} g_{\alpha,b} \bar{R}_{\alpha,\beta} \text{ conj.}
\]
Since \( \det | g_{\alpha,b} | \neq 0 \) and \( \det | g^{ab} | \neq 0 \), we have the lemma.

Lemma 5. In a fibred Hermitian space \( B(X,Y) \) with a field \( F \) with lifts satisfying \( A_1 \)-condition whose fibres are isometric, the Christoffel symbols \( \{ \alpha_j \} \) vanish if and only if the quantities \( G_{\alpha,\beta} \) vanish.

Proof. \[
\{ \alpha_j \} = \frac{1}{2} \left( g^{ab} g_{\alpha,b} - g^{ab} g_{\beta,a} + g^{ac} g_{\alpha,c} - g_{\alpha,c} \right)
\]
\[
= \frac{1}{2} g^{ab} \left( g_{\alpha,b} \bar{R}_{\alpha,\beta} - F_{j,k} \Gamma_{\alpha}^j \right)
\]
\[
= \frac{1}{2} g^{ab} G_{\alpha,\beta}.
\]
Since \( \det | g^{ab} | \neq 0 \), we have the lemma.

§ 3. Fibred Kählerian space. In this section let us consider some analogous propositions concerning with \( \{ 5 \} \) in a fibred Kählerian space. If a fibred Hermitian space satisfies Kähler's condition\(^3\), the space is called a fibred Kählerian space.

Now we can easily show the following proposition by Lemma 1, 4 and 5.

Proposition 2. In a fibred Kählerian space with a field \( F \) with lifts satisfying \( A \)-condition whose fibres are isometric, the field \( F \) is completely integrable and its fibres are parallel.

Next we shall prove the following main theorem as an analogue of Theorem 4

\(^3\) For the Christoffel symbols \( \{ \alpha_j \} = 0 \) (conj.) or for the Hermitian metric \( \frac{\partial g_{\alpha,\beta}}{\partial \tau} = \frac{\partial g_{\alpha,\beta}}{\partial \bar{\tau}} \) (conj.).
of Y. Mutô (5).

**Theorem.** In a fibred Kählerian space $B(X,Y)$ with a field $F$ with lifts satisfying $A$-condition whose fibres are isometric, $B$ is decomposed locally into the product $X \times Y$.

Proof. By virtue of Lemma 4, in a fibred Kählerian space $B(X,Y)$ with a field $F$ with lifts satisfying $A$-condition whose fibres are isometric, the field $F$ is completely integrable. Hence the quantities $\Gamma^a_a$ (conj.) and $g_{i\alpha}$ (conj.) vanish in a set of favourable coordinate system (Lemma in § 1). In such a set of favourable coordinate system we have

$$G_{ab} = g_{ab} \text{; conj.}.$$

Again by Proposition 1,

$$g_{ab,\xi} = G_{ab,\xi} \equiv 0, \quad g_{\xi\iota,\xi} = G_{\xi\iota,\xi} \equiv 0.$$

Hence we can identify $g_{ab}$'s to components of the metric tensor in $X$.

Since fibres are isometric, we have

$$F_{\xi;a} = 0, \quad F_{\xi;\xi} = 0.$$  

Moreover in a set of favourable coordinate system such that the quantities $\Gamma^\xi_a$ (conj.) and $g_{i\alpha}$ (conj.) vanish identically, the above equations become

$$F_{\xi;a} = g_{\xi,i} a = 0, \quad F_{\xi;\xi} = g_{\xi,i} a = 0.$$

This shows that the quantities $g_{\xi,i}$'s do not contain $z^a$ and $\bar{z}^a$. By the first equations of (4) we can identify $g_{\xi,i}$'s to components of the metric tensor in $Y$.

Therefore we can conclude that in the set of favourable coordinate system under consideration

$$ds^2 = g_{ij} (dz^i d\bar{z}^j + g_{ab} dx^a d\bar{z}^a),$$

i.e., $B$ is decomposed locally into the product $X \times Y$. q.e.d.

We have the following corollary which is an example of Theorem in T. Saeki (7).

**Corollary.** A Riemannian connection in a fibred Kählerian space $B(X,Y)$ with a field $F$ with lifts satisfying $A$-condition whose fibres are isometric, is completely reducible.

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