COMPLETE ORTHOGONAL DECOMPOSITION HOMOMORPHISMS BETWEEN MATRIX ORDERED HILBERT SPACES

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Abstract. The purpose of this paper is to show that a complete order homomorphism and a complete orthogonal decomposition homomorphism between the non-commutative \( L^2 \)-spaces induce respectively an isomorphism and a \( * \)-isomorphism between the associated reduced von Neumann algebras.

1. Introduction

In [C] Connes studied an order isomorphism on a Hilbert space and introduced an orientable homogeneous selfdual cone to construct a von Neumann algebra. On the other hand, Schmitt and Wittstock [SW] introduced a matrix ordered Hilbert space to handle a non-commutative order and characterized it using the face property of the family of selfdual cones. From the point of view of the complete positivity of the maps on a matrix ordered Hilbert space, we showed in [M2] the relationship between an order isomorphism or an orthogonal decomposition isomorphism defined by Yamamuro [Y] and an isomorphism of a von Neumann algebra. In the present article we shall generalize their results to the case where a complete order homomorphism is not necessarily bijective.

We shall use the notation as introduced in [SW] with respect to the matrix ordered standard forms.

Let \( M_n \) and \( M_{n,m} \) be respectively a set of all \( n \times n \) and \( n \times m \) matrices over \( \mathbb{C} \). For a Hilbert space \( H \) and \( n \in \mathbb{N} \), put \( H_n = H \otimes M_n \). Let \( (H, H_n^+, n \in \mathbb{N}) \), where \( H_n^+ \) denotes a selfdual cone in \( H_n \), be a matrix ordered Hilbert space, and let \( (\hat{H}, \hat{H}_n^+, n \in \mathbb{N}) \) be another one. Let \( h \) be a bounded linear map of \( H \) into \( \hat{H} \). A bijective linear map \( h \) is called an order isomorphism if \( hH_n^+ = \hat{H}_n^+ \). We call \( h \) a complete order isomorphism if \( hH_n^+ = \hat{H}_n^+ \) for every \( n \in \mathbb{N} \). We call \( h \) an o.d. (orthogonal decomposition) homomorphism if \( h \) is 1-positive and \( (h\xi, h\eta) = 0 \) whenever \( \xi, \eta \in H^+ \) and \( (\xi, \eta) = 0 \). If \( h_n \) is an o.d. homomorphism for every \( n \in \mathbb{N} \), we call \( h \) a complete o.d. homomorphism. A bijective map \( h \) is called a complete o.d. isomorphism if both \( h \) and \( h^{-1} \) are complete o.d. homomorphisms.

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From now on, let \((M, H, H^+_n, n \in \mathbb{N})\) and \((\hat{M}, \hat{H}, \hat{H}^+_n, n \in \mathbb{N})\) be matrix ordered standard forms of von Neumann algebras. Here we use the notation

\[ \text{Ad}(h) : x \in M \mapsto h x h^{-1} \in B(\hat{H}) \]

for the invertible map \(h : H \rightarrow \hat{H}\).

Throughout this paper, we assume a Hilbert space to be separable.

2. Results

The main results are as follows:

**Theorem A.** Let \((M, H, H^+_n, n \in \mathbb{N})\) and \((\hat{M}, \hat{H}, \hat{H}^+_n, n \in \mathbb{N})\) be matrix ordered standard forms. Suppose that \(h\) is a complete order homomorphism of \(H\) into \(\hat{H}\) with support projection \(e\) and range projection \(f\). Put \(N = M \cap \{e\}'\) and \(\hat{N} = M \cap \{f\}'\). If \(e\) is completely positive and \(h H^+\) is a selfdual cone in the closed range space of \(h\), then we obtain the following properties:

1. \(f\) is completely positive.
2. \((e M|_{eH}, e H, e_n H_n^+, n \in \mathbb{N})\) and \((f \hat{M}|_{f\hat{H}}, f \hat{H}, f_n \hat{H}_n^+, n \in \mathbb{N})\) are matrix ordered standard forms.
3. \(h|_{eH}\) is a complete order isomorphism of \(eH\) onto \(f\hat{H}\), and \(\text{Ad}(h|_{eH})\) is an isomorphism of \(e M_e\) onto \(f \hat{M}_f\).

**Theorem B.** With \((M, H, H^+_n, n \in \mathbb{N})\) and \((\hat{M}, \hat{H}, \hat{H}^+_n, n \in \mathbb{N})\) as before, let \(h\) be a completely positive o.d. homomorphism of \(H\) into \(\hat{H}\) with support projection \(e\) and range projection \(f\). If \(h\) has the closed range, then we obtain the following properties:

1. \(e\) belongs to \(M \cap M'\).
2. \(f\) is completely positive.
3. \((f \hat{M}|_{f\hat{H}}, f \hat{H}, f_n \hat{H}_n^+, n \in \mathbb{N})\) is a matrix ordered standard form.
4. \(h|_{eH}\) is a complete a.d. isomorphism of \(eH\) onto \(f\hat{H}\), and \(\text{Ad}(h|_{eH})|_{M_e}\) is a \(*\)-isomorphism of \(M_e\) onto \(f \hat{M}_f\).

We need some lemmata to prove Theorem A.

**Lemma 1.** Let \((M, H, J, H^+)\) be a standard form, and let \(\hat{H}\) be a Hilbert space with a selfdual cone \(\hat{H}^+\). Suppose that \(h\) is a linear bijection of \(H\) onto \(\hat{H}\) such that \(h H^+ = \hat{H}^+\). Then, for the polar decomposition \(h = u|h|\) of \(h\), \(u\) is a 1-positive isometry of \(H\) onto \(\hat{H}\), and there exists a positive invertible operator \(k\) in \(M\) such that \(|h| = k J k J\).

**Proof.** Since for every \(\xi \in \hat{H}^+, (h^* \xi, \eta) = (\xi, h^* \eta) \geq 0\) holds for all \(\eta \in H^+\), it follows from the selfduality of \(H^+\) that \(h^* H^+ \subset H^+\). Hence \(h^* h H^+ \subset H^+\). Since \((h^{-1})^* = (h^*)^{-1}\), \(h^* h H^+ = H^+\). By \([\text{[3] Theorem 3.3]}\) there exists a positive invertible operator \(k\) in \(M\) such that \(h^* h = k^2 J k^2 J\). Note that we may assume \(H^+ = \mathcal{P}_0\) with a cyclic and separating vector \(\xi_0 \in \hat{H}^+\) by the unicity of the standard form. Then \(|h| = k J H^+ k J H^+\), and

\[ u H^+ = h k^{-1} J k^{-1} J H^+ = h H^+ = \hat{H}^+. \]

This completes the proof. \(\square\)
Lemma 2. With \((M, H, H^+_n, n \in \mathbb{N})\) a matrix ordered standard form and \((\hat{H}, \hat{H}^+_n, n \in \mathbb{N})\) a matrix ordered Hilbert space, let \(h\) be an order isomorphism of \(H\) onto \(\hat{H}\). If \(h\) is completely positive, then \(h\) is a complete order isomorphism. In addition, there exists a von Neumann algebra \(M\) such that \((M, H, H^+_n, n \in \mathbb{N})\) is a matrix ordered standard form, and \(\text{Ad}(h)\) is an isomorphism of \(M\) onto \(\hat{M}\).

Proof. Let \(h = u|h|\) be the polar decomposition of \(h\). By Lemma 1, \(|h|\) can be written as \(|h| = kJ_{H^+}kJ_{H^+}\) for some positive invertible operator \(k \in M\). Hence

\[
|h_n|H^+_n = (k \otimes 1_n)J_{H^+}H^+_n (k \otimes 1_n)J_{H^+}H^+_n = H^+_n.
\]

Then \(u_nH^+_n = h_nH^+_n \subset \hat{H}^+_n\). Since \(u\) is unitary, \(u\) is a complete order isomorphism of \(H\) onto \(\hat{H}\). Thus we see that \(h\) is a complete order isomorphism and \(u\) is a complete o.d. isomorphism. By [M2], Proposition 2.6, Theorem 2.7 we obtain the desired result.

Lemma 3. With \((M, H, H^+_n, n \in \mathbb{N})\) a matrix ordered standard form, let \(e\) be a completely positive projection on \(H\). Then there exists a von Neumann algebra \(A\) such that \((A, eH, e_nH^+_n, n \in \mathbb{N})\) is a matrix ordered standard form. In addition, if \(N = M \cap \{e\}'\), then

\[
A = eM|eH = N|eH.
\]

Proof. Put \(J = J_{H^+}, K = eH, K_n = e_nH_n, K^+ = eH^+, K^+_n = e_nH^+_n\) for every \(n \in \mathbb{N}\). There exists by [M1], Lemma 1 a von Neumann algebra \(A\) such that \((A, K, K^+)\) is a matrix ordered standard form. The inclusion \(eM|K \subset A\) follows from the first part of the proof of [M1] Lemma 2. We prove that \(A \subset N|K\). Note that in a standard form \((M, H, J, H^+)\) the map \(q \mapsto qJqH^+\) is an order isomorphism of the set of all projections in \(M\) onto the set of all closed faces in \(H^+\) (see [C] Theorem 4.2 c)). If \(p\) is a projection in \(A\), then \((0, 0)^T J_{K^+} J_{K_2^+} \Xi = P_F \Xi = P_{(F)} \Xi = P_{(F')} \Xi\) for all \(\Xi \in K_2^+\). By setting \(\Xi = (0, 0)^T\) we have

\[
\begin{bmatrix}
0 \\
0
\end{bmatrix}

= b_Jb_J \xi_c + b_Jc_J \xi_c
\]

for all \(\xi \in K\). It follows from [SW] Corollary 3.3 that \(b_J\xi = 0\) for all \(\xi \in K\). Using both equalities \(\xi = c_Jc_J \xi\) and \(b_Jb_J + c^2 = c\) since \(P\) is a projection, we have

\[
c_J \xi = c_Jc_J \xi = (c^2 b_J) \xi = c_Jc_J \xi = \xi.
\]

Moreover, by setting \(\Xi = (0, 0)^T\) we have

\[
p_J \xi = e_Jc_J \xi
\]

for all \(\xi \in K\). Therefore, when \(\xi\) is an element of \(K^+\), \(p_J \xi = e_Jc_J \xi = e_J \xi\). Hence, \(p_J \xi = e_J \xi\) for all \(\xi \in K\) because \(K\) is spanned by \(K^+\). Since \(e_2P_{(F')} = P_{(F')}e_2\),
ea = ae. Consequently, we obtain

$$N|_{K} \subset eM|_{K} \subset A \subset N|_{K}.$$

Therefore, we obtain the desired equalities. \(\square\)

**Remark.** In the above lemma, if we assume that \((eH, e_nH^+_n, n \in \mathbb{N})\) is a matrix ordered Hilbert space having the conditions for a matrix ordered standard form and \(e\) is 2-positive instead of the complete positivity, then Lemma 3 holds. Namely, if there exists a matrix ordered standard form \((A, K, K^+_n, n \in \mathbb{N})\) and \(eH^+ = K^+, e_2H^+_2 = K^+_2\), then \(A = eM|_{K} = N|_{K}\).

**Proof of Theorem A.** Put \(K = eH, K^+ = eH^+, K^+_n = e_nH^+_n, \hat{K} = hH, \hat{K}^+ = hH^+\) and \(\hat{K}^+_n = h_nH^+_n\). Let \(h = u|h|\) and \(h_0 = u_0|h_0|\) be the polar decompositions of \(h\) and \(h_0 = h|_{K}\), respectively. Since \(e\) is a completely positive projection, it follows from Lemma 3 that \((eM|_{K}, K, K^+_n)\) is a matrix ordered standard form.

By assumption \(h_0\) is an order isomorphism of \(K\) onto \(\hat{K}\). Hence by Lemma 2 \(|h_0|\) is a complete order isomorphism on \(K\). Therefore, \(|h_0|nK^+_n\) is a selfdual cone in \(K_n\), and so is \(\hat{K}^+_n\) in \(K_n\) because of the complete positivity of \(h\). Since \(f_n\) is the support projection of \(h_n\), it follows that \(\hat{K}^+_n \subset f_nH^+_n\). If \(\xi \in \hat{K}^+_n, \eta \in H^+_n\), then \((\xi, f_n\eta) = (\xi, \eta) \geq 0\). Hence \(\hat{K}^+_n \subset (f_nH^+_n)'\) (in \(K_n\)). Therefore, \(\hat{K}^+_n = \hat{K}^+ + (f_nH^+_n)' \supset f_nH^+_n\) (in \(K_n\)). Hence \(\hat{K}^+ = f_nH^+_n\), which means that \(f\) is completely positive. Therefore, by Lemma 3 \((fM|_{K_n}, \hat{K}, \hat{K}^+_n)\) is a matrix ordered standard form and \(\text{Ad}(h_0)\) is an isomorphism of \(eM|_{K}\) onto \(fM|_{K}\). \(\square\)

Now, we examine the properties of o.d. homomorphisms between two ordered Hilbert spaces.

**Proposition 4 (cf. [DY, (2.1)]).** Let \((M, H, J, H^+)\) be a standard form, and let \(\hat{H}\) be a Hilbert space with a selfdual cone \(\hat{H}^+\). Then \(h\) is an o.d. homomorphism of \(H\) into \(\hat{H}\) if and only if \(hH^+ \subset \hat{H}^+\) and \(|h| \in M \cap M'\).

One can give the similar proof to that of Dang-Yamamuro.

**Proposition 5 (cf. [DY, (3.1)]).** With \((M, H, J, H^+)\) and \(\hat{H}, \hat{H}^+\) as before, if \(h\) is a bijective o.d. homomorphism of \(H\) to \(\hat{H}\), then \(h\) is an o.d. isomorphism.

**Proof.** Let \(h = u|h|\) be the polar decomposition of \(h\). Using the argument in the proof of Proposition 4, we see that \(h\) is an order isomorphism. By Lemma 1, \(|h|\) can be written as \(|h| = kJkJ\) for some positive invertible operator \(k \in M\). Since \(u = h^{-1}k^{-1}Jk^{-1}J\), it follows that \(uH^+ \subset \hat{H}^+\). Hence \(u\) is an o.d. homomorphism, and so \(u\) is an o.d. isomorphism. Hence \(|h|\) is an o.d. homomorphism. By [Y (3.4)], \(k\) belongs to \(M \cap M'\). This means that \(|h|^{-1}\) is an o.d. homomorphism. Therefore, \(h\) is an o.d. isomorphism. This completes the proof. \(\square\)

**Lemma 6.** With \((M, H, H^+_n, n \in \mathbb{N})\) a matrix ordered standard form and \((\hat{H}, \hat{H}^+_n, n \in \mathbb{N})\) a matrix ordered Hilbert space, let \(h\) be a completely positive o.d. homomorphism of \(H\) into \(\hat{H}\). Then \(h_nH^+_n\) is a selfdual subcone of \(H^+_n\) and \((hH^+, h_nH^+_n, n \in \mathbb{N})\) is a matrix ordered Hilbert space, and \(h\) is a complete o.d. homomorphism.

**Proof.** Let \(h = u|h|\) be the polar decomposition of \(h\). Using Proposition 4, we see that \(|h|\) belongs to \(M \cap M'\). Hence \(|h_n|\) belongs to \(M_n \cap M'_n\). This implies that \(h_n\) is an o.d. homomorphism, i.e., \(h\) is a complete o.d. homomorphism. To complete
the proof, it suffices to show that $h_n H_n$ is selfdual; hence $|h_n| H_n$ is selfdual. Recall that for every $n \in \mathbb{N}$ the selfdual cone $H_n$ is generated by the elements $\{x_i, x_j H + \xi\}_{i,j=1}^n$, $x_1, \ldots, x_n \in M, \xi \in H^+$. If we set $h_\varepsilon = |h| + \varepsilon 1, \varepsilon > 0$, then for such elements $x_i, \xi$, we get

$$
\lim_{\varepsilon \to 0} |h_n| \left[ h_\varepsilon^{-\frac{1}{2}} x_i H + h_\varepsilon^{-\frac{1}{2}} x_j H + \xi_1 \right]_{i,j=1}^n = \lim_{\varepsilon \to 0} \left[ |h| h_\varepsilon^{-1} x_i H + x_j H + \xi_1 \right]_{i,j=1}^n
$$

where $e$ denotes the support projection of $|h|$ and it belongs to the center of $M$. This implies that $|h_n| H_n^+$ is dense in the selfdual cone $e_n H_n^+$.

Proof of Theorem B. We apply Proposition 4, Proposition 5, Lemma 6 and [M2, Theorem 2.7].

Applying Lemma 3 and Theorem B, we obtain the following corollary:

**Corollary 7.** For matrix ordered standard forms $(M, H, H_n^+, n \in \mathbb{N})$ and $(\hat{M}, \hat{H}, \hat{H}_n^+, n \in \mathbb{N})$, suppose that $u$ is a completely positive partial isometry with initial projection $e$ and final projection $f$. Put $\rho(x) = xu^*$ for all $x \in e Me$. Then $(e M|e H, e H, e_n H_n^+, n \in \mathbb{N})$ and $(f \hat{M}|f \hat{H}, f \hat{H}, f_n \hat{H}_n^+, n \in \mathbb{N})$ are matrix ordered standard forms, and $\rho$ is a $*$-isomorphism of $e Me$ onto $f M f$.

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