PAPER Special Section on Discrete Mathematics and Its Applications

Efficient Enumeration of All Ladder Lotteries with k Bars

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SUMMARY A *ladder lottery*, known as the "Amidakuji" in Japan, is a network with n vertical lines and many horizontal lines each of which connects two consecutive vertical lines. Each ladder lottery corresponds to a permutation. Ladder lotteries are frequently used as natural models in many areas. Given a permutation π, an algorithm to enumerate all ladder lotteries of π with the minimum number of horizontal lines is known. In this paper, given a permutation π and an integer k, we design an algorithm to enumerate all ladder lotteries of π with exactly k horizontal lines.

key words: *algorithm, enumeration, ladder lottery, family tree*

1. Introduction

A *ladder lottery*, known as the "Amidakuji" in Japan, is a common way to choose an assignment randomly. Formally, a ladder lottery of a permutation $\pi = (p_1, p_2, \ldots, p_n)$ is a network with n vertical lines (*lines* for short) and many horizontal lines (*bars* for short) as follows. The i-th line from the left is called *line* i. The top ends of the n lines correspond to π. The bottom ends of the n lines correspond to the identity permutation $(1, 2, \ldots, n)$. Each bar connects two consecutive lines. Each number p_i in π starts at the top end of line i, and goes down along the line, then whenever p_i comes to an end of a bar, p_i goes horizontally along the bar to the other end, then goes down again. Finally p_i reaches the bottom end of line p_i. We can regard a bar as a modification of the "current" permutation. In a ladder lottery a sequence of such modifications always results in the identity permutation $(1, 2, \ldots, n)$. Figure 1 shows a ladder lottery of permutation $(2, 6, 4, 1, 5, 3)$. It consists of six lines and fourteen bars. For example, number 6 starts at the top end of line 2, and finally reaches the bottom end at line 6. For each bar in the figure, two exchanged numbers are written.

The ladder lotteries are strongly related to primitive sorting networks, which are deeply investigated by Knuth [2]. A comparator in a primitive sorting network replaces p_i and p_j by $\min(p_i, p_j)$ and $\max(p_i, p_j)$, while a bar in a ladder lottery always exchanges them.

Given a permutation $\pi = (p_1, p_2, \ldots, p_n)$ the minimum number of bars to construct ladder lotteries of π is equal to the number of "inversions" in π, which are pairs (p_i, p_j) in π with $p_i > p_j$ and $i < j$. A ladder lottery of π with the minimum number of bars is *optimal*. The ladder lottery in Fig. 1 is non-optimal, since its correspondence permutation (2,6,1,5,3) has eight inversions: (2,1), (6,4), (6,1), (6,5), (6,3), (4,1), (4,3) and (5,3). The ladder lottery of the same permutation in Fig. 2 is optimal, since the number of bars is eight. Optimal ladder lotteries appear in a variety of areas [6]: algebraic combinatorics and computational geometry, etc. On the other hand, "non-optimal" ladder lotteries are very typical for Japanese. When Japanese use a ladder lottery to assign roles or duties to members in a group, a ladder lottery with many bars is preferred than an optimal one. The reason is to enjoy an assignment game using the ladder lottery with many bars.

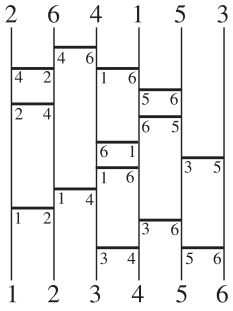

Fig. 1 A ladder lottery of the permutation (2,6,4,1,5,3) with 14 bars.

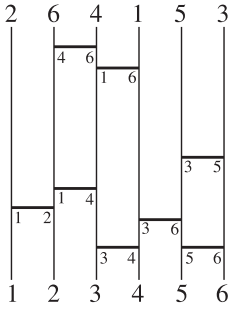

Fig. 2 An optimal ladder lottery of the permutation (2,6,4,1,5,3).

In [6] we gave an algorithm to enumerate all optimal ladder lotteries of a given permutation π. The algorithm generates all optimal ladder lotteries of π in $O(1)$ time for each. The idea of our algorithm in [6] is as follows. We first define a tree structure T_π, called *the family tree*, among optimal ladder lotteries of π, in which each vertex of T_π corresponds to each optimal ladder lottery and each edge of T_π corresponds

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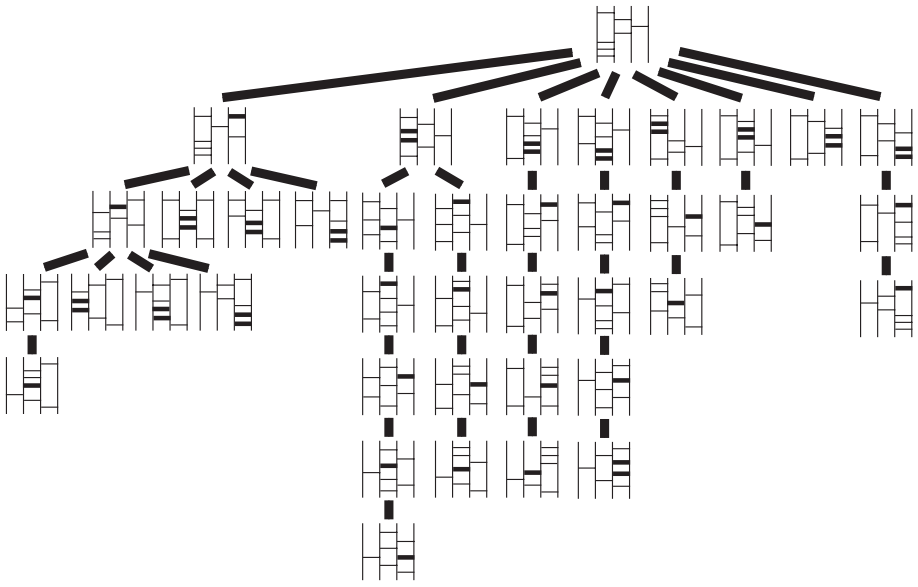


Fig. 3 The family tree $T_{\pi,k}$, where $\pi = (4, 2, 3, 1)$ and $k = 7$.

to a relation between two optimal ladder lotteries. Then we design an efficient algorithm to generate all child vertices of a given vertex in T_π. Applying the algorithm recursively from the root vertex of T_π, we can generate all vertices in T_π, and also corresponding optimal ladder lotteries. Based on such tree structure, but with some other ideas, a lot of efficient enumeration algorithms are designed [1], [3], [4].

The algorithm in [6] works only if the number of bars is minimum. In this paper we generalize the algorithm to enumerate all ladder lotteries in $S_{\pi,k}$, which is the set of all ladder lotteries of a given permutation π with exactly k bars. To define the family tree, we adopted swap operations (which will be defined in Sect. 2) as a relation between two optimal ladder lotteries in [6]. However, we cannot define a (spanning) tree structure among $S_{\pi,k}$ using swap operations, because there may exist two ladder lotteries L, L' in $S_{\pi,k}$ such that L is not derived from L' by swap operations. Therefore we first introduce warp operations (which will be defined in Sect. 2); then we show that a tree structure, called the family tree, among $S_{\pi,k}$ can be defined (see Fig. 3); finally we design an enumeration algorithm. The algorithm enumerates all ladder lotteries in $S_{\pi,k}$ in $O(1)$ time for each. Note that if k is smaller than the number of inversions in π then $S_{\pi,k} = \phi$. Also if the parity of k does not match the parity of the number of inversions in π then $S_{\pi,k} = \phi$. In this paper we design a new family tree (see Fig. 3) to enumerate all ladder lotteries in $S_{\pi,k}$ for any k. By using our enumeration algorithm, we can generate the catalog of all ladder lotteries in $S_{\pi,k}$. It would be useful to investigate some properties of ladder lotteries in $S_{\pi,k}$.

The rest of the paper is organized as follows. Section 2 gives some definitions. Section 3 defines the tree structure among ladder lotteries in $S_{\pi,k}$. Section 4 gives an efficient algorithm to enumerate all ladder lotteries in $S_{\pi,k}$. Finally Sect. 5 is a conclusion.

2. Preliminary

A *ladder lottery* L of a permutation $\pi = (p_1, p_2, \ldots, p_n)$ is a network with n vertical lines (*lines* for short) and many horizontal lines (*bars* for short) as follows. The i-th line from the left is called *line* i. The top ends of the n lines correspond to π. The bottom ends of the n lines correspond to the identity permutation $(1, 2, \ldots, n)$. Each bar connects two consecutive vertical lines. See Fig. 1. Each number p_i in π starts the top end of line i, and goes down along the line, then whenever p_i comes to an end of a horizontal bar, p_i goes to the other end, then goes down again. Finally p_i reaches the bottom end of line p_i. This path is called *the* p_i*-route*. We can regard a bar as a modification of the "current" permutation. In a ladder lottery a sequence of such modifications always results in the identity permutation $(1, 2, \ldots, n)$. The p_i-route is *x-monotone* if p_i always goes right along a bar on p_i-route. Note that p_i-route with no bar is x-monotone. Intuitively p_i goes right-down on x-monotone p_i-route.

Let $\pi = (p_1, p_2, \ldots, p_n)$ be a permutation. An *inversion* in π is a pair (p_i, p_j) with $p_i > p_j$ and $i < j$. Let r be the number of inversions in π. If a ladder lottery L contains exactly r bars, then we say that L is *optimal*.

A *swap operation*, which corresponds the notion of "braid relation" in the area of algebra, is a local modification of a ladder lottery as shown in Fig. 4. Note that each dashed circle contains exactly three bars. Applying this modification to a ladder lottery of π resutls in other ladder lottery of π, since the local permutation consisting of the modified three bars remains as it was. A swap operation (a) to (b) in Fig. 4 is called a *left swap operation to bar* b_u. Note that in Fig. 4 the left swap operation moves bar b_u from the (upper) right of the 5-route to the (lower) left, and to apply the left swap operation we need some route, say the 3-route, to be

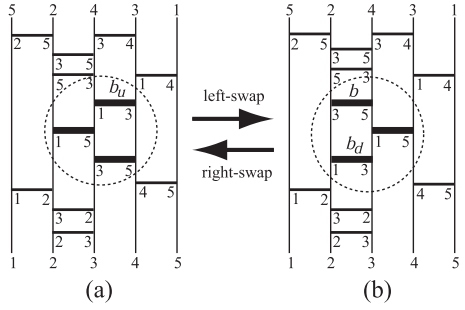


Fig. 4 A local swap operation.

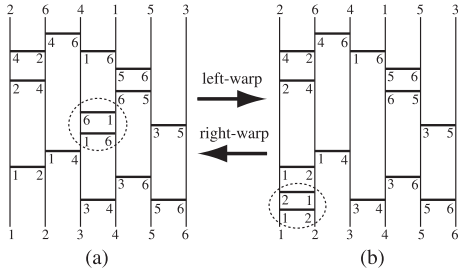


Fig. 5 A warp operation.

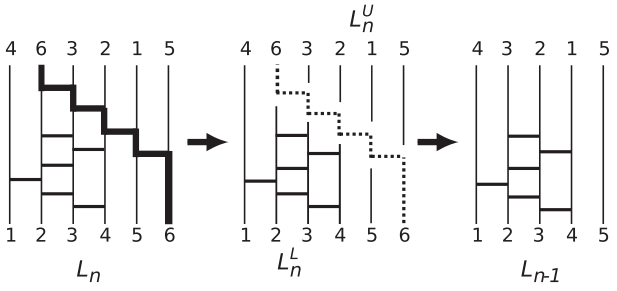


Fig. 6 The removal of the n-path.

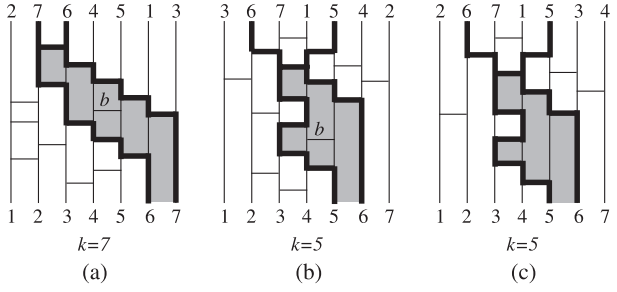


Fig. 7 The active regions.

left-turned there. Similarly, a local swap operation (b) to (a) in Fig. 4 is called a *right swap operation to bar* b_d. Note that the operation moves bar b_d from the left of the 5-route to the right.

A *redundant pair* is a pair of parallel bars appearing consecutively between the same pair of lines. In Fig. 5 examples are shown in the dashed circles. A *left warp operation* is a modification of a ladder lottery as shown in Fig. 5, in which (1) remove some redundant pair, then (2) append it at the lower left corner. We define a right warp operation as follows. Let L be a ladder lottery, and b be a redundant pair of L. Let L' be a ladder lottery derived from L by a left warp operation to b. Then a *right warp operation* is (1) remove b in L', then (2) append it so that L is again derived from L'. Intuitively, a right warp operation is the reverse operation of a left warp operation.

3. The Family Tree

Let $S_{\pi,k}$ be the set of all ladder lotteries of a given permutation $\pi = (p_1, p_2, \ldots, p_n)$ with exactly k bars. Assume that k is not smaller than the number of inversions in π, and the parity of k and the number of inversions match.

In this section we design a tree structure $T_{\pi,k}$ among $S_{\pi,k}$, in which each vertex of $T_{\pi,k}$ corresponds to a ladder lottery in $S_{\pi,k}$, and each edge corresponds to a relation between two ladder lotteries.

Assume $S_{\pi,k} \neq \phi$. Pick up any $L_n \in S_{\pi,k}$. Observe the n-route in L_n. Recall n-route is a path in which n in π goes from the top end of line p_i with $p_i = n$ to the bottom end of line n. The n-route partitions L_n into the upper part L_n^U and the lower part L_n^L. We say L_n is *n-clean* if (i) L_n^U has no bar and (ii) the n-route is x-monotone. If L_n is n-clean then removing the n-route from L_n then patching L_n^U and L_n^L, as shown in Fig. 6, results in a ladder lottery, say L_{n-1}, in $S_{\pi',k-1}$, where π' is the permutation derived from π by removing n. Similarly observe the $(n-1)$-route in L_{n-1}. We say L_{n-1} is *$(n-1)$-clean* if (i) L_{n-1}^U has no bar and (ii) the $(n-1)$-route is x-monotone. Repeat this process until some non-clean ladder lottery appears or L_2 is derived. If L_i is i-clean for each $i = 3, 4, \ldots, n$, then L_n is called *the root lottery* of $S_{\pi,k}$, denoted by R (See Figs. 8 and 12). Otherwise we define the clean level of L_n as follows. *The clean level* of L_n is c if L_i is i-clean for $i = n, n-1, \ldots, c$ but not $(c-1)$-clean. Especially if L_n is not n-clean then the clean level of L_n is $n+1$, and the clean level of R is 3. Note that if $L_n \in S_{\pi,k}$ has the clean level c, then n-route, $(n-1)$-route, $\ldots$, c-route form so called "a brick structure," as follows.

For each $p_a \geq c$ in π, let $(q_1, q_2, \ldots, q_b)$ be the decreasing list of numbers each of which is larger than p_a and locating to the left of p_a in π. In L_n, the p_a-route first go left b times, along the bars sharing with q_1-route, q_2-route, $\ldots$, q_b-route, then turn, then go right $p_a - a + b$ times. Note that on the right side of the p_a-route with $p_a \geq c$, every x-route with $x < c$ always goes left along a bar, every bar is on the y-route for some $y > p_a$, and there is no bar to which a left swap operation can be applied. Otherwise there exists some route with left-turn, a contradiction. Also either (1) L_n has at least one bar in the region below the c-route and above the $(c-1)$-route, or (2) the $(c-1)$-route is not x-monotone in L_{c-1}. See some examples in Fig. 7. The region is called *the active region* of L_n. We define the active region of R is ϕ for convenience (in the proof of Lemma 2).

Now we assign the *parent ladder lottery* in $S_{\pi,k}$ for each ladder lottery L_n in $S_{\pi,k} \setminus \{R\}$ as follows. We assume that L_n has the clean level c. Let AP (Active Path) be the maximal x-

monotone subpath of the $(c-1)$-route ending at the bottom of line $c-1$. We say a bar b connecting line l and $l+1$ is *upward visible from AP* if (1) the lowest end of a bar on l above AP is the end of b, and (2) the lowest end of a bar on $l+1$ above AP is the end of b. Note that if b is upward visible from AP then b can be left-swapped and other upward visible bar from AP never has an end on line l nor $l+1$. Thus the number of the upward visible bars from AP is at most $\frac{n}{2}$. Now we define the parent ladder lottery of $L_n \in S_{\pi,k} \setminus \{R\}$, as follows. We have the following two cases.

Case 1: The active region has at least one visible bar from AP.

Among the upward visible bars from AP, the rightmost bar is called *the active bar* of L_n. In Figs. 7(a) and (b), b is the active bar. Apply the left swap operation to the active bar and let $P(L_n)$ be the derived ladder lottery.

Case 2: The active region has no visible bar from AP.

Then at the left end of AP there exists a redundant pair, which is called *the active pair*. See Fig. 7(c). Apply the left warp operation to the active pair and let $P(L_n)$ be the derived ladder lottery.

We say $P(L_n)$ is *the parent ladder lottery* of L_n, and L_n is a *child ladder lottery* of $P(L_n)$. Note that the parent ladder lottery of L_n is unique, while $P(L_n)$ may have many children. Also note that the clean level of $P(L_n)$ is smaller than or equal to the clean level of L_n, and if they have the same clean level then $P(L_n)$ has less or equal number of bars in the active region, and if they have the same number of bars in the active region then $P(L_n)$ has shorter AP.

We have the following lemma.

Lemma 1: For any $L_n \in S_{\pi,k} \setminus \{R\}$, $P(L_n) \in S_{\pi,k}$ holds.

Proof. Since the each of the two operations to derive the parent preserves the permutation. *Q.E.D.*

Given a ladder lottery L_n in $S_{\pi,k} \setminus \{R\}$, by repeatedly finding the parent ladder lottery of the derived ladder lottery, we can have the unique sequence $L_n, P(L_n), P(P(L_n)), \ldots$ of ladder lotteries in $S_{\pi,k}$, which eventually ends up with the root lottery R. See Fig. 8. The active bars and the active pairs are depicted by thick lines.

We have the following lemma.

Lemma 2: The sequence $L_n, P(L_n), P(P(L_n)), \ldots$ of $L_n \in S_{\pi,k} \setminus \{R\}$ ends with $R \in S_{\pi,k}$.

Proof. For each $L_n \in S_{\pi,k}$ we define *its clean potential* $C(L_n) = (s,t,u)$, where s is the clean level of L_n, t is the number of bars in the active region of L_n and u is the length of AP. For $L_1, L_2 \in S_{\pi,k}$ with $C(L_1) = (s_1,t_1,u_1)$ and $C(L_2) = (s_2,t_2,u_2)$, we say L_1 *is cleaner than* L_2 if (1) $s_1 < s_2$, (2) $s_1 = s_2$ and $t_1 < t_2$, or (3) $s_1 = s_2$, $t_1 = t_2$ and $u_1 < u_2$. For any $L_n \in S_{\pi,k}$ we can observe that $P(L_n)$ is cleaner than L_n and R is the cleanest among $S_{\pi,k}$. Thus for any $L_n \in S_{\pi,k}$ the sequence of clean potentials $C(L_n), C(P(L_n)), C(P(P(L_n))), \ldots$ always ends at $C(R)$. *Q.E.D.*

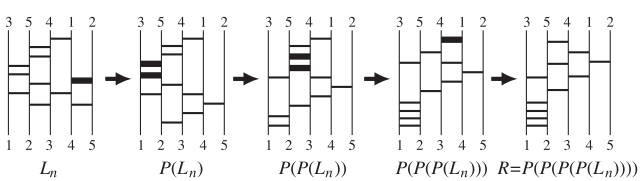


Fig. 8 The sequence of a ladder lottery L_n of (3,5,4,1,2) with exactly 11 bars.

By merging all these sequences we can have *the family tree* of $S_{\pi,k}$, denoted by $T_{\pi,k}$, in which the root vertex of $T_{\pi,k}$ corresponds to R, the vertices of $T_{\pi,k}$ correspond to the ladder lotteries in $S_{\pi,k}$ and each edge corresponds to a relation between a ladder lottery in $S_{\pi,k}$ and its parent. See Fig. 3. The active bars and the active pairs are depicted by thick lines.

4. Enumerating All Ladder Lotteries

In this section we give an efficient algorithm to enumerate all ladder lotteries in $S_{\pi,k}$.

If we have an algorithm to enumerate all children of a given ladder lottery in $S_{\pi,k}$, then by recursively applying the algorithm starting at the root lottery R of $S_{\pi,k}$, we can enumerate all ladder lotteries in $S_{\pi,k}$. Now we design such an algorithm.

We need some definitions. Let $L_n \in S_{\pi,k}$ with $\pi = (p_1, p_2, \ldots, p_n)$. Assume L_n has the clean level c. So each bar locating on the right of the c-route is contained in some x-route with $x > c$, but either (1) in the active region (See Fig. 7) there is at least one bar which is not contained in any x-route with $x \geq c-1$ or (2) the $(c-1)$-route is not x-monotone. Each x-route with $x \geq c$ goes left along bars (sharing with larger routes), "turns," then goes right along bars (sharing with smaller routes). For each x-route with $x \geq c$, if b is the first bar to go right after bars to go left, then b is called *the turn bar* of x. Note that only if the x-route contains both at least one bar to left and one bar to right, the x-route has the turn bar. Otherwise if the x-route contains only bars to left (or right) then the x-route has no turn bar in L_n. Also note that the turn bar is defined only for the x-route with $x \geq c$. In the next lemma we show that on the x-route with $x \geq c$, only turn bars has a chance to be right swapped.

Lemma 3: Let L_n be a ladder lottery having the clean level c. On the x-route with $x \geq c$ only the turn bar has a chance to be right swapped.

Proof. Since L_n has the clean level c, the x-route of each $x \geq c$ first goes left along bars, turns, then goes right along bars. A bar b_d can be right swapped only if the vertical segment between the left end of b_d and the left end of b has no right end of other bars, where b is the lowermost bar among the bars above b_d as shown in Fig. 4(b). Here the route passes b, then left-turn, then passes b_d. Thus on the x-route with $x \geq c$, only the turn bar has a chance to satisfy

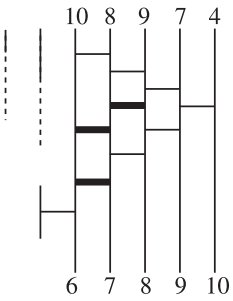


Fig. 9 Illustration for Lemma 3.

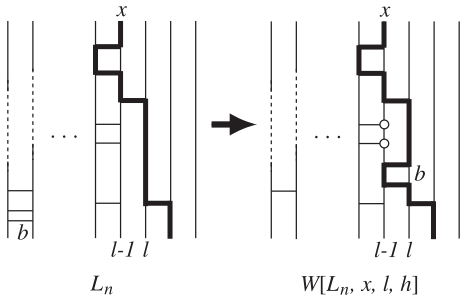


Fig. 10 Illustration for $W[L_n, x, l, h]$.

the condition. See an example in Fig. 9. The turn bars of the 9-route and the 7-route can be right swapped, but the turn bar of the 8-route can not be. *Q.E.D.*

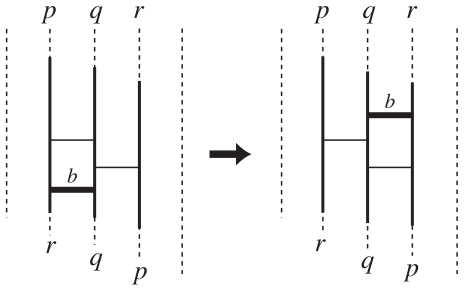
Let $S[L_n, b]$ be the ladder lottery derived from L_n by applying the right swap operation to a bar b. If $L_n \in S_{\pi,k}$ has the clean level c and has a redundant pair b at the lower left corner of L_n then let $W[L_n, x, l, h]$ be the ladder lottery derived from L_n by a right warp operation to b so that (1) b has ends at line $l-1$ and l, (2) b is on the x-route of $W[L_n, x, l, h]$, (3) b appears at the left end of AP of $W[L_n, x, l, h]$, and (4) the vertical segment between the left end of b and the left end of the bar on the x-route above b has exactly h right ends of bars in $W[L_n, x, l, h]$. See Fig. 10, where $h = 2$ corresponds the two white circles. Every child of L_n is either $S[L_n, b]$ for some b or $W[L_n, x, l, h]$ for some x, l and h, but not all $S[L_n, b]$ or $W[L_n, x, l, h]$ are children of L_n. $S[L_n, b]$ is a child of L_n only if b is the active bar of $S[L_n, b]$. Also $W[L_n, x, l, h]$ is a child of L_n only if b is the active pair of $W[L_n, x, l, h]$.

Now we first classify each $S[L_n, b]$ into children of L_n and non-children, as follows. Remember the clean level of L_n is c. Let $R(i)$ be the region on the right side of the i-route, and $L(i)$ be the region on the left side of the i-route. We have two types.

Type 1: b is a turn bar.

If b can not be right swapped then $S[L_n, b]$ is not defined. So we assume otherwise.

Note that such a bar exists only on the x-routes with $x \geq c$. We assume that b is on the q-route, and in L_n the routes of p, q, r pass through as shown in Fig. 11(a). Since



(a) L_n (b) $S[L_n, b]$

Fig. 11 Illustration for Type 1.

b is a turn bar of the q-route, $q \geq c$ and $p > q > r$ hold. Now $S[L_n, b]$ is not p-clean since b in $S[L_n, b]$ is not on the x-route with $x > p$. Thus the clean level of $S[L_n, b]$ is increased to $p + 1$, and b is the only bar in the active region of $S[L_n, b]$. Since b is upward visible from AP, which is the maximal x-monotone subpath of the p-route ending at the bottom of line p, b is the active bar of $S[L_n, b]$. Thus $L[L_n, b]$ is a child of L_n. Otherwise $S[L_n, b]$ is not a child of L_n.

Type 2: b can be right swapped but b is not a turn bar.

Such a bar, say b, exists only in $L(c) \cap L(c+1) \cap \cdots \cap L(n)$, and b is "downward" visible from some x-route, and a right swap operation to b moves b to $R(x)$ crossing the x-route.

Note that the left boundary of $L(c) \cap L(c+1) \cap \cdots \cap L(n)$ is x-monotone. If the right swap operation moves b to $R(x)$ crossing the x-route with $x \geq c$, then the clean level of $S[L_n, b]$ is $x + 1$ and b is the only bar in the new active region and b is upward visible from AP, so b is the active bar of $S[L_n, b]$. Thus $S[L_n, b]$ is a child of L_n.

If the right swap operation moves b to $R(c-1)$, crossing AP, which is the maximal x-mononotone subpath of the $(c-1)$-route ending at the bottom of line $c - 1$, then the clean level of $S[L_n, b]$ remains c, and b is appended to the active region. Assume the active bar of L_n has the left end on line s and b in L_n has the right end on line t. So b in $S[L_n, b]$ has the right end on line $t + 1$. If $t + 1 \geq s$, then b is the active bar of $S[L_n, b]$, otherwise b is not. Thus $S[L_n, b]$ is a child of L_n if and only if $t + 1 \geq s$.

If the right swap operation moves b to $R(c-1)$, crossing the $(c-1)$-route but not AP, then b is not upward visible from AP. Thus $S[L_n, b]$ is not a child of L_n.

Otherwise the right swap operation moves b to $R(x)$ crossing the x-route with $x < c - 1$. The clean level of $S[L_n, b]$ remains c, and b is not the active bar in $S[L_n, b]$. Thus $S[L_n, b]$ is not a child of L_n.

Also we classify each $W[L_n, x, l, h]$ into children of L_n and non-children, as follows.

Type 3: $W[L_n, x, l, h]$.

If L_n has no redundant pair at the lower left corner of L_n then $W[L_n, x, l, h]$ is not defined. Assume otherwise. Let

Algorithm 1: `enumerating-ladder-lotteries` $(\pi = (p_1, p_2, \ldots, p_n), k)$

```
1  begin
2      Construct the root lottery R in S_{π,k}
3      for each x ≥ 2 do
4          if the turn bar b of the x-route is right swappable then
5              find-all-children(S[R, b], x + 1, b)
6      if L_n has no redundant pair at the lower left corner then
7          return
8      for each x ≥ 2 do
9          for each l = max(u_x, 2) to x do
10             for each possible h do
11                 find-all-children(W[R, x, l, h], c, φ)
```

c be the clean level of L_n.

Each x-route with $x \geq c$ goes left along bars, "turns," then goes right along bars. For each $x \geq c$, let u_x be the leftmost line on which the x-route passes. Note that AP is x-monotone.

Now for each $x \geq c$ and each l, where $x \geq l \geq \max(u_x, 2)$, and each possible h, $W[L_n, x, l, h]$ is a child of L_n. For l with $x \geq l > u_x$, $W[L_n, x, l, h]$ is derived from L_n by removing a redundant pair at the lower left corner, then replace a suitable bar on the x-route by triple bars. For $l = u_x$, $W[L_n, x, l, h]$ is derived from L_n by removing a redundant pair at the lower left corner, then appending a redundant pair so that the redundant pair appears at the left end of AP of $W[L_n, x, l, h]$.

For $x = c-1$ we need to check more carefully. Assume AP of L_n has left end on line u. If L_n has no active bar, then for each l with $c - 1 \geq l \geq \max(u, 2)$ and each possible h, $W[L_n, c-1, l, h]$ is a child of L_n.

If L_n has the active bar b, then assume it has the left end on line s. Note that we have $u < s$. Then for each l with $c - 1 \geq l \geq s$ and each possible h, $W[L_n, c-1, l, h]$ is a child of L_n. Note that the left end of AP of $W[L_n, c-1, l, h]$ is on line $l - 1$. For each l with $s - 1 \geq l \geq \max(u, 2)$, $W[L_n, c-1, l, h]$ still has the active bar b, so the parent of $W[L_n, c-1, l, h]$ is not L_n, and $W[L_n, c-1, l, h]$ is not a child of L_n.

For $x < c - 1$, $W[L_n, x, l, h]$ is not a child of L_n.

By the above case analysis, we have the algorithm as shown in Algorithms 1 and 2. Algorithm 1 is the main routine of our algorithm. First we construct the root lottery R in $S_{\pi,k}$, where π is a given permutation and k is a given integer. Then we generate all children of R and call a procedure Algorithm 2 for each child. Algorithm 2 generates all children of a given ladder lottery.

By maintaining (1) the clean level c, (2) the list of downward visible bar from x-route, for each $x \geq c - 1$ (those are candidate bars to be right swapped, crossing the x-route), (3) active path AP, (4) the active bar, (5) the maximal x-monotone subpath of the x-route ending at the bottom of line x for each $x \geq c$ (We append redundant pairs along these paths), and (6) the current ladder lottery, we can enumerate all children of L_n in $O(1)$ time for each on average.

Algorithm 2: `find-all-children`(L_n, c, a)

```
1  begin
       /* L_n is the current ladder lottery, c is the
          clean level of L_n, and a is the active bar
          or the active pair, and s is the line on
          which a has the left end.                    */
2
3      Output L_n    /* Output the difference from the
          previous one.                                */
4
5      for each x ≥ c do
6          if the turn bar b of the x-route is right swappable then
7              find-all-children(S[L_n, b], x + 1, b)
8          for each downward visible (non-turn) bar b from the
           x-route do
9              find-all-children(S[L_n, b], x + 1, b)
10     for each downward visible bar b from AP do
           /* b in L_n has the right end on line t.
              */
11         if t ≥ s − 1 then
12             find-all-children(S[L_n, b], c, b)
13     if L_n has no redundant pair at the lower left corner then
14         return
15     for each x ≥ c do
           /* u_x is the leftmost line on which the
              x-route passes.                          */
16         for each l = max(u_x, 2) to x do
17             for each possible h do
18                 find-all-children(W[L_n, x, l, h], x + 1, φ)
19     if L_n has no active bar then
           /* u is the line on which AP of L_n has
              left end.                                */
20         for each l = max(u, 2) to c − 1 do
21             find-all-children(W[L_n, x, l, h], c, φ) for a
               suitable h
22     else
23         for each l = s to c − 1 do
24             find-all-children(W[L_n, x, l, h], c, φ) for a
               suitable h
```

Lemma 4: One can enumerate all children of L_n in $O(1)$ time for each.

Proof. Let L_n be the current ladder lottery, and assume that its clean level is c and AP is its active path of L_n. We show that, given (1)–(6), we can compute each child and update (1)–(6) for the child in $O(1)$ time as follows.

Each of Type 1 child and Type 2 child is generated by a bar in the lists of (2), and each Type 3 child is generated by removing a redundant pair, then appending a redundant pair to a suitable place along paths in (5) and along active path from active bar.

Now we explain how to update data structures as in the following three cases.

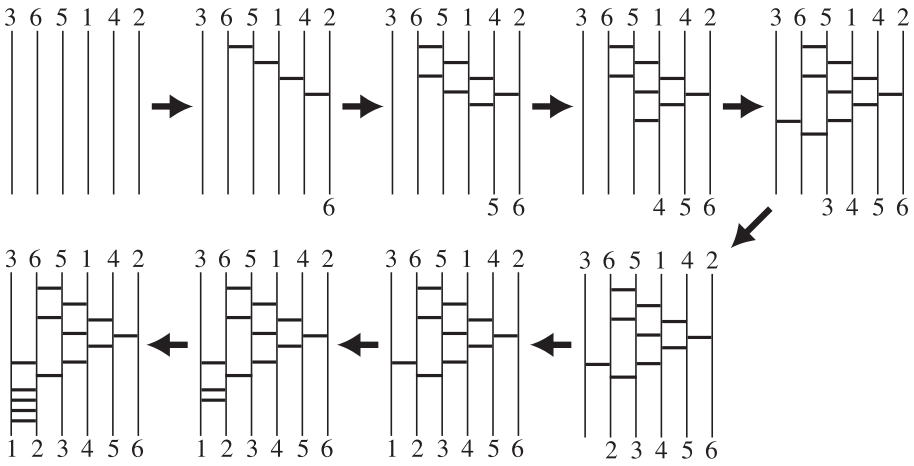


Fig. 12 Computation of the root lottery R in $S_{\pi,k}$, where $\pi = (3, 6, 5, 1, 4, 2)$ and $k = 14$.

Case 1: Type 1 child $S[L_n, b]$.

Now b is the turn bar of the y-route with $y \geq c$, and assume b is downward visible from x-route with $x \geq c$. (1) The clean level becomes to be $x + 1$. (2) Assume b is downward visible from the z-route with $z > x$ in $S[L_n, b]$. If the turn bar b_x of x-route is downward visible from z-route, then we insert b as the next bar of b_x in the list of z-route. Note that there is no downward visible bar from z-route in the "left" of b in $S[L_n, b]$. Also note that, if b_x becomes to be downward non-visible from z-route by swapping b, we exchange b_x with b in the list of z-route. Otherwise, b_x is downward non-visible from z-route, then we insert b as the first bar into the list of z-route. Note that there is no downward visible bar from z-route in the "left" of b in $S[L_n, b]$. Therefore, we can update these in $O(1)$ time. (3) If $x \geq c$, then active path is updated with the maximal x-monotone subpath of x-route. (4) The active bar updated with b in $S[L_n, b]$. (5) The paths of w-route for $w \geq x + 1$ in $S[L_n, b]$ remain as the ones of w-route in L_n. (6) $S[L_n, b]$ can be constructed from L_n in $O(1)$ time.

Case 2: Type 2 child $S[L_n, b]$.

Now b is not a turn bar. Assume b is downward visible from x-route with $x \geq c$. If $x \geq c$ holds, then similar to Case 1, we can update (1)–(6) in $O(1)$ time. Otherwise, $x = c - 1$, (1) the clean level of $S[L_n, b]$ remains as c. (2) The list of downward visible bars from x-route for each $x \geq c$ is similar to Case 1. The list for $x = c - 1$ in $S[L_n, b]$ is derived from the list of downward visible bars from $(c - 1)$-route in L as follows. Replace the bars up to b by at most two bars each of which is downward visible from $(c - 1)$-route in $S[L_n, b]$ but not in L_n. Thus we can compute (2) in $S[L_n, b]$ in $O(1)$ time. (3) The active path remains as AP. (4) The active bar updated with b in $S[L_n, b]$. (5) The paths of w-route for $w \geq c$ in $W[L_n, x, l, h]$ remain as the ones of w-route in L_n. (6) $S[L_n, b]$ can be constructed from L_n in $O(1)$ time.

Case 3: Type 3 child $W[L_n, x, l, h]$.

(1) The clean level becomes to be $x+1$, and (2) the lists in L_n remain as ones in $W[L_n, x, l, h]$. (3) The active path is updated with a subpath of x-route of L_n from line l to the bottom end of x-route of L_n. (4) $W[L_n, x, l, h]$ has no active bar. (5) The paths of w-route for $w \geq x + 1$ in $W[L_n, x, l, h]$ remain as the ones of w-route in L_n. (6) $W[L_n, x, l, h]$ can be constructed from L_n in $O(1)$ time. Note that one can determine whether or not L_n contains redundant pairs in the lower left corner in $O(1)$ time.

Q.E.D.

From the above lemma, we obtain the following theorem.

Theorem 1: After constructing and outputting the root lottery R in $S_{\pi,k}$ in $O(n+k)$ time, the algorithm runs in $O(|S_{\pi,k}|)$ time. The algorithm uses $O(n + k)$ working space.

Proof. We show that R in $S_{\pi,k}$ can be generated in $O(n+k)$ time. See Fig. 12 for a sketch. We start with n vertical lines. Then we append the j-route for each $j = n, n-1, \ldots, 3$. Each j-route goes left with some bars, turns, then goes right with some bars. When we append the j-route the part of route to go left is already completed, since those bars correspond to the crossing with the routes of larger numbers. So we only need to append the part to go right, consisting of x-monotone path. Finally we append the redundant pairs at the lower left corner. Thus we can compute R in $O(n + k)$ time and space. *Q.E.D.*

By the theorem above, our algorithm generates each ladder lottery in $S_{\pi,k}$ in $O(1)$ time "on average." However it may have to return from the deep recursive calls without outputting any ladder lottery in $S_{\pi,k}$, after generating a ladder lottery corresponding to the rightmost leaf of a large subtree in the family tree. Therefore the next ladder lottery in $S_{\pi,k}$ cannot be generated in $O(1)$ time in worst case.

By modifying the algorithm so that each ladder lottery at "even" depth in $T_{\pi,k}$ is output "before" its children, and each ladder lottery at "odd" depth in $T_{\pi,k}$ is output "after" its children [5], we can output the next ladder lottery in $O(1)$ time in worst case.

Theorem 2: After constructing and outputting the root lottery R in $S_{\pi,k}$ in $O(n+k)$ time, the algorithm enumerates all ladder lotteries in $S_{\pi,k}$ in $O(1)$ time for each. The algorithm uses $O(n+k)$ working space.

5. Conclusion

In this paper, we gave an algorithm to enumerate all ladder lotteries of a given permutation π with exactly k bars. Our algorithm uses $O(n+k)$ space and enumerate all ladder lotteries in $S_{\pi,k}$ in $O(1)$ time for each in worst case.

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