Ann. Rep. Fac. Edu., Iwate Univ., Vol.72 (March, 2013), 93 ~ 99

On Prescribing Curvature of Contact 3-Manifolds *

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(Received January 8, 2013)

Abstract

Recently, V. Krouglov studied prescribing curvatures of contact closed 3-manifolds (M, ω), and proved, among others, that if Ker ω is trivial then for any smooth function on M there is a Riemannian metric of M so that the sectional curvature of Ker ω coincides with the given function. In this paper, we replace this triviality condition of Ker ω to curvature conditions, and show similar results.
Keywords: contact structure, prescribing curvature, k-admissible metric.

1 Introduction

Let (M, ω) be a contact closed 3-manifold. Set $\xi = \text{Ker}\,\omega$. Recently, V. Krouglov [4] studied prescribing curvatures of contact 3-manifolds, and proved that if ξ is trivial then for any smooth function f on M there is a Riemannian metric of M so that the sectional curvature $K(\xi) = f$. In this paper, we change this triviality condition of ξ to curvature conditions, that is, we show that, if there is a suitable metric, then for any smooth function on M there is a Riemannian metric of M so that the sectional curvature of ξ coincides with the given function. Though V. Krouglov began his argument with an arbitrary metric of M, we shall begin the argument with a kind of adapted metrics introduced by S. Chern and R. Hamilton [2].

We shall give preliminaries and auxiliary results in §2, and present and prove main results in §3. In §4, we give some examples.

2 Preliminaries and auxiliary results

In this paper, we work in the C^∞-category. In what follows, we always assume that a contact structure is given by a one-form ω, and that the ambient manifold M is closed, connected, oriented and of dimension 3, unless otherwise stated (see [1], [3], [5] for the generalities on contact structures).

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* 2000 *Mathematics Subject Classication.* 53C12.

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Let $\xi = \operatorname{Ker} \omega$, and assume that $\omega \wedge d\omega > 0$ on M. We can find a Riemannian metric g of M satisfying the following: Let be a unit vector field orthogonal to ξ with $\omega(\ ) = 1$, $\iota_N d\omega = 0$ and $\omega \wedge d\omega = kdV(M, g)$, where k is a positive constant and $dV(M, g)$ is the volume element of the Riemannian manifold (M, g). We call such a metric k-adapted for ω. Note that in case $k = 2$, that is, $\omega \wedge d\omega = 2dV(M, g)$, g is called adapted in [3]. Thus, if we give an oriented local orthonormal frame $\{\ , E, F\}$ with $E, F \in \Gamma(\xi)$, then we have $\omega \wedge d\omega = kdV(M, g) = k\ {}^* \wedge E^* \wedge F^*$, where $V^* = g(V,\ )$. By definition, $d\omega/k$ is the volume element along ξ. It follows that ξ is minimal in the sense that

$$\langle \nabla_E \ , E\rangle + \langle \nabla_F \ , F\rangle = 0,$$

where ∇ is the Riemannian connection of (M, g).

We locally represent covariant derivatives explicitly (see [6] for details). Let U be an open subset of M.

Proposition 1. Let $\{\ , E, F\}$ be an oriented orthonormal frame on U. As $\nabla\ E \perp E,\ $, we can set $\nabla\ E = \rho F$ for a smooth function ρ on U. Set also $\langle \nabla_E E,\ \rangle = \lambda$ and $\langle \nabla_E E,\ \rangle = \alpha$. Then we have

(0) $\langle [E, F],\ \rangle = -k$.

(1) $\nabla\ \ = 0$, $\nabla\ E = \rho F$, $\nabla\ F = -\rho E$.

(2) $\nabla_E\ = -\lambda E - \alpha F$, $\nabla_E E = -\operatorname{div}(F)F + \lambda N$, $\nabla_E F = \operatorname{div}(F)E + \alpha N$.

(3) $\nabla_E\ = -(k + \alpha)E + \lambda F$, $\nabla_F E = \operatorname{div}(E)F + (k+\alpha)\ $, $\nabla_F F = -\operatorname{div}(E)E - \lambda N$.

Define the second fundamental form B of ξ by

$$\mathrm{B(V,W)} = \frac{1}{2}\langle \nabla_V W + \nabla_W V,\ \rangle$$

for all sections V, W of ξ (cf. [4], [7]). This is a symmetric bilinear form on ξ. As ξ is minimal, it follows that Tr B = 0. Define also the *extrinsic curvature* $K_e(\xi)$ by

$$K_e(\xi) = \frac{B(V,V)B(W,W) - B(V,W)^2}{\langle V,V\rangle\langle W,W\rangle - \langle V,W\rangle^2},$$

where V, W are two linearly independent sections of ξ (cf. [4]). Note that $K_e(\xi) \leq 0$ because Tr B $= B(E,E) + B(F,F) = 0$ for an oriented orthonormal basis $\{E, F\}$ of ξ. By using Proposition 1, we have the following (see [6] for details).

Proposition 2. Let $\{\ , E, F\}, U, B$ be as above. At $p \in U$, $B_p = 0$ if and only if $Ric(\ ,\ )_p = k^2/2$. In particular, $B = 0$ on U if and only if is a Killing vector field on U. In this case, we have $\lambda = 0$ and $\alpha = -k/2$.

3 Main results and proofs

Our main results are the following.

Theorem 1. Let ω be a contact structure on an oriented, connected, and closed 3-manifold M. Set $\xi = \mathrm{Ker}\ \omega$. If there is a k-adapted Riemannian metric for ω satisfying $K_M > -3k^2/4$, where K_M is the sectional curvature of (M, g), then, for any smooth function f on M, there is a Riemannian metric of M so that $K(\xi) = f$.

Theorem 2. Let ω be a contact structure on an oriented, connected, and closed 3-manifold M. Set $\xi = \mathrm{Ker}\ \omega$. If there is a k-adapted Riemannian metric for ω satisfying $Ric_M < k^2/2$, then for any smooth function f on M there is a Riemannian metric of M so that $K(\xi) = f$. In particular, if there is a k-adapted Riemannian metric for ω satisfying $K_M < k^2/4$, then for any smooth function f on M there is a Riemannian metric of M so that $K(\xi) = f$.

Note that by Proposition 2, we always have $Ric(\ ,\ ) \le k^2/2$. Note also that as argued in §7.2 in [1], k-admissible metrics cannot be of strictly negative curvature.

We prove these results by the same way as V. Krouglov [4]. The difference is that, instead of the triviality of ξ, we assume the existence of a k-adapted Riemannian metric g for ω having the property $K(\xi) > -3k^2/4$ or $Ric(\ ,\ ) < k^2/2$.

Let (M, ω, g) be as above and $U \subset M$ be an open set. Let   be the unit vector field orthogonal to $\xi =$ Ker ω with $\omega(\ ) = 1$, $\iota\ d\omega = 0$, $\omega \wedge d\omega = k dV(M, g)$ and $\{\ , E, F\}$ be an oriented local orthonormal frame with $E, F \in \Gamma(\xi)$ on U. Let φ be a positive smooth function on M. Define a new metric g of M by

$$g = \frac{1}{\varphi^2} g|_{\xi^{\perp}} \oplus g|_{\xi}.$$

By definition, $\bar{\ } = \varphi N$ is the unit vector field orthogonal to ξ with respect to this metric $\bar{g}$, and $\{\bar{\ }, E, F\}$ is an oriented local orthonormal frame with respect to $\bar{g}$. Set $g(X, Y) = \langle X, Y \rangle$ and $\bar{g}(X, Y) = \langle X, Y \rangle_{\varphi}$ for $X, Y \in \Gamma(TM)$. Recall the Koszul formula for the Riemannian connection ∇ of $\langle\ ,\ \rangle$:

$$2\langle \nabla_S T, U\rangle = S\langle T, U\rangle + T\langle U, S\rangle - U\langle S, T\rangle + \langle [S, T], U\rangle - \langle [S, U], T\rangle - \langle [T, U], S\rangle$$

for $S, T, U \in \Gamma(TM)$.

Using this formula, we get the following by simple calculations:

(1) $\bar{\nabla}_F F = \langle \nabla_F F, E\rangle E + \varphi \langle \nabla_F F,\ \rangle \bar{\ }$,

(2) $\bar{\nabla}_E F = \langle \nabla_E F, E\rangle E + (\varphi B(E, F) - k/(2\varphi)) \bar{\ }$,

(3) $\bar{\nabla}_E \bar{\ } = (k/(2\varphi) - \varphi B(E, F))F - \varphi \langle \nabla_E E,\ \rangle E$,

(4) $\bar{\nabla}_F \bar{\ } = -(k/(2\varphi) + \varphi B(E, F))E - \varphi \langle \nabla_F F,\ \rangle F$,

(5) $\overline{\nabla}_F E = \langle \nabla_F E, F\rangle F + (\varphi B(E,F) + k/(2\varphi))\bar{\ }$,

(6) $\overline{\nabla}_{\bar{\ }} F = -\varphi \langle \nabla\ E, F\rangle E + k(\varphi - 1/\varphi)E/2 - F(\log\varphi)\bar{\ }$.

Using these formulas in this order, we have the following.

Proposition 3. For the sectional curvature $\overline{K}(\xi)$ along ξ with respect to $\bar{g}$ we have

$$\overline{K}(\xi) = K(\xi) - \varphi^2 K_e(\xi) + B(E,E)B(F,F) - \langle \nabla_E F, \nabla_F E\rangle + \frac{k^2}{2} - \frac{3k^2}{4\varphi^2}.$$

Proof. Let $\{\ , E, F\}$ be an oriented orthonormal frame with respect to g and $\{\bar{\ }, E, F\}$ be an oriented orthonormal frame with respect to $\bar{g}$ on U. It follows that

$$\begin{aligned}
\overline{K}(\xi) &= \overline{K}(E,F) = \langle \overline{R}(E,F)F, E\rangle_\varphi \\
&= \langle \overline{\nabla}_E \overline{\nabla}_F F, E\rangle_\varphi - \langle \overline{\nabla}_F \overline{\nabla}_E F, E\rangle_\varphi - \langle \overline{\nabla}_{[E,F]} F, E\rangle_\varphi \\
&= \langle \overline{\nabla}_E (\langle \nabla_F F, E\rangle E + \varphi \langle \nabla_F F, \ \rangle \bar{\ }), E\rangle_\varphi \\
&\quad - \langle \overline{\nabla}_F \left(\langle \nabla_E F, E\rangle E + \left(\varphi B(E,F) - \frac{k}{2\varphi}\right)\bar{\ }\right), E\rangle_\varphi \\
&\quad - \langle \overline{\nabla}_E F, E\rangle_\varphi \langle \overline{\nabla}_E F, E\rangle_\varphi + \langle \overline{\nabla}_F E, F\rangle_\varphi \langle \overline{\nabla}_F F, E\rangle_\varphi \\
&\quad - \frac{1}{\varphi^2}\langle [E,F], \bar{\ }\rangle \langle \overline{\nabla}_{\bar{\ }} F, E\rangle_\varphi \\
&= E\langle \nabla_F F, E\rangle - \varphi^2 \langle \nabla_F F, \ \rangle \langle \nabla_E E, \ \rangle \\
&\quad - F\langle \nabla_E F, E\rangle + \left(\varphi B(E,F) - \frac{k}{2\varphi}\right)\left(\varphi B(E,F) + \frac{k}{2\varphi}\right) \\
&\quad - \langle \nabla_E E, F\rangle^2 - \langle \nabla_F F, E\rangle^2 + \frac{k}{\varphi}\left(-\varphi \langle \nabla\ E, F\rangle + \frac{k}{2}\left(\varphi - \frac{1}{\varphi}\right)\right) \\
&= K(\xi) - \varphi^2 K_e(\xi) + B(E,E)B(F,F) - \langle \nabla_E F, \nabla_F E\rangle + \frac{k^2}{2} - \frac{3k^2}{4\varphi^2}.
\end{aligned}$$

Proof of Theorem 1. Let g be a k-adapted Riemannian metric with the property $K(\xi) + 3k^2/4 > 0$ and f be an arbitrary smooth function on M. We can choose a small positive constant C so that $K(\xi) + 3k^2/4 - Cf > 0$ on M, because M is compact. Let $\{\ , E, F\}$ be an oriented orthonormal frame of $U \subset M$. Then, for the metric $\bar{g} = \frac{1}{\varphi^2} g|_{\xi^\perp} \oplus g|_\xi$ we have

$$\overline{K}(\xi) = K(\xi) - \varphi^2 K_e(\xi) + B(E,E)B(F,F) - \langle \nabla_E F, \nabla_F E\rangle + \frac{k^2}{2} - \frac{3k^2}{4\varphi^2}$$

by Proposition 3. It is easy to see that the right hand side formula is independent of the choices of oriented orthonormal frames on U. Thus, the following equation on φ is globally defined one over M:

$$Cf = K(\xi) - \varphi^2 K_e(\xi) + B(E,E)\,B(F,F) - \langle \nabla_E F, \nabla_F E\rangle + \frac{k^2}{2} - \frac{3k^2}{4\varphi^2}.$$

If we could get a positive solution φ of this equation, we would have proved our Theorem.

Set $t = 1/\varphi^2$ and at each point $x \in M$ consider the quadratic equation on t:

$$\frac{3k^2}{4}t^2 + \left(Cf - K(\xi) - B(E,E)\,B(F,F) + \langle \nabla_E F, \nabla_F E\rangle - \frac{k^2}{2}\right)t + K_e(\xi) = 0.$$

Note that, as $K_e(\xi) \leq 0$, the discriminant D of this equation satisfies

$$D = \left(Cf - K(\xi) - B(E,E)\,B(F,F) + \langle \nabla_E F, \nabla_F E\rangle - \frac{k^2}{2}\right)^2 - 3k^2 K_e(\xi) \geq 0.$$

It follows that this equation has a non-negative solution at each point $x \in M$. We have to show that this solution is positive and smooth on M. To see this, consider the case $K_e(\xi) = 0$. Note that the solution is positive and smooth at the points $x \in M$ with $K_e(\xi)\,(x) < 0$. In case $K_e(\xi) = 0$, by Proposition 2, we have $\lambda = 0$ and $\alpha = -k/2$, that is, the coefficient of t becomes

$$Cf - K(\xi) - B(E,E)\,B(F,F) + \langle \nabla_E F, \nabla_F E\rangle - \frac{k^2}{2} = Cf - K(\xi) - \frac{3k^2}{4} < 0,$$

which shows that the solution is positive and smooth, too. By a conformal change of this metric using C, we get the desired metric $\tilde{g}$ with $\tilde{K}(\xi) = f$ and this completes the proof.

Proof of Theorem 2. From the above proof, it is easy to see that if $K_e(\xi) < 0$ then we get the same result as in Theorem 1. As $K_e(\xi) \leq 0$, it is sufficient to show that $K_e(\xi) \neq 0$.
By Proposition 2, we have, at $p \in M$

$$K_e(\xi)_p = 0 \Leftrightarrow B_p = 0 \Leftrightarrow Ric(\ ,\ )_p = \frac{k^2}{2}.$$

By the assumption that $Ric_M < k^2/2$, we have $K_e(\xi) < 0$, which completes the proof of Theorem 2.

4 Examples

To apply our theorem, we have just to note that $K(\xi) + 3k^2/4 - Cf > 0$.

The first example is due to [5]. Let $(\mathbf{R}^3, g_0)$ be the 3-dimensional Euclidean space with the canonical coordinate (x, y, z). Define $\omega = \sin z dx + \cos z dy$. Then, it follows that

$$d\omega = \cos z dz \wedge dx - \sin z dz \wedge dy, \text{ and } \omega \wedge d\omega = dx \wedge dy \wedge dz = dV(\mathbf{R}^3, g_0).$$

This is 1-adapted. Define $\quad = \sin z\partial_x + \cos z\partial_y$, $E = \partial_z$ and $F = \cos z\partial_x - \sin z\partial_y$. Then, it is easy to see that $\{\ \ , E, F\}$ is an oriented orthonormal frame on $\mathbf{R}^3$. We get covariant derivatives

$$\nabla \quad = 0,\ \nabla \ \ E = 0,\ \nabla \ \ F = 0,$$
$$\nabla_E \quad = \cos z\partial_x - \sin z\partial_y,\ \nabla_E E = 0,\ \nabla_E F = -\sin z\partial N_x - \cos z\partial_y,$$
$$\nabla_F \quad = 0,\ \nabla_F E = 0,\ \nabla_F F = 0.$$

Thus, we have

$$\rho = \langle \nabla \ \ E, F\rangle = 0,\ \lambda = \langle \nabla_E E, \quad \rangle = 0,\ \alpha = \langle \nabla_E F, \quad \rangle = -1,$$

and

$$B(E, E) = B(F, F) = 0, B(E, F) = B(F, E) = -1/2,$$
$$K(\xi) = 0, K_e(\xi) = -1/4.$$

For an arbitrary smooth function f on $\mathbf{R}^3$, the equation becomes

$$f = K(\xi) - \varphi^2 K_e(\xi) + B(E, E)\, B(F, F) - \langle \nabla_E F, \nabla_F E\rangle + \frac{1}{2} - \frac{3}{4\varphi^2}$$
$$= \frac{\varphi^2}{4} + \frac{1}{2} - \frac{3}{4\varphi^2},$$

whose positive solution is

$$\varphi = \sqrt{2f - 1 + \sqrt{(2f-1)^2 + 3.}}$$

Thus, if we want to get $\overline{K}(\xi) = 1$ then take $\varphi = \sqrt{3}$, and if we want to get $\overline{K}(\xi) = -1$ then take $\varphi = \sqrt{2\sqrt{3} - 3}$. Note that on the flat torus T^3, we get the same conclusion because the above quantities are well defined on T^3.

The second one is due to [2]. Let (S^3, g_1) be the unit sphere in the 4-dimensional Euclidean space $\mathbf{R}^4$ with the canonical coordinate (x, y, z, w). Define a 1-form ω by

$$\omega = xdy - ydx + zdw - wdz.$$

It is easy to see that $\omega \wedge d\omega = 2dV(S^3, g_1)$, thus $k = 2$. Set $\quad = x\partial_y - y\partial_x + z\partial_w - w\partial_z$, $E = x\partial_z - z\partial_x - y\partial_w + w\partial_y$ and $F = x\partial_w - w\partial_x - y\partial_z - z\partial_y$. Then $\{\ \ , E, F\}$ is an oriented orthonormal frame of S^3 with $\omega(\ \ ) = 1$ and $E, F \in \mathrm{Ker}\,\omega$. We get covariant derivatives

$$\nabla \quad = 0,\ \nabla \ \ E = -F,\ \nabla \ \ F = E,$$
$$\nabla_E E = 0,\ \nabla_E F = - \quad ,\ \nabla_E \quad = F,$$
$$\nabla_F E = \quad ,\ \nabla_F F = 0,\ \nabla_F \quad = -E.$$

Thus, we have

$$\rho = \langle \nabla \ \ E, F\rangle = -1,\ \lambda = \langle \nabla_E E, \quad \rangle = 0,\ \alpha = \langle \nabla_E F, \quad \rangle = -1,$$

and

$$B(E, E) = B(F, F) = B(E, F) = B(F, E) = 0.$$

Note that Ker ω is non-trivial. For a smooth function f on S^3 and positive constant C with $Cf < 4$, the equation becomes

$$Cf = K(\xi) - \varphi 2K_e(\xi) + B(E,E)\,B(F,F) - \langle \nabla_E F, \nabla_F E\rangle + \frac{2^2}{2} - \frac{3\ \ 2^2}{4\varphi^2}$$
$$= 1 - 0 + 0 + 1 + 2 - \frac{3}{\varphi^2}$$
$$= 4 - \frac{3}{\varphi^2}$$

whose positive solution is

$$\phi = \sqrt{\frac{3}{4 - Cf}}.$$

Thus, if we want to get $\overline{K}(\xi) = 0$ then take $\varphi = \sqrt{3/4}$, and if we want to get $\overline{K}(\xi) = -1$ then take $\varphi = \sqrt{3/5}$.

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