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Admissible Vector Fields of Codimension-One Plane Fields and Their Stability Property

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Abstract

Given a codimension-one plane field ξ on a closed manifold M, we show that if X is transverse to ξ, then there are many functions f on M such that $\mathrm{supp}(f) = M$ and that fX is the mean curvature vector field of ξ with respect to some Riemannian metric of M, and we give a necessary and sufficient condition for X to become the mean curvature vector field of ξ with respect to some Riemannian metric of M. As an application, we show a stability property of mean curvature vector fields H of ξ with respect to small perturbations of codimension-one plane fields.

1 Introduction

Let M be an oriented closed manifold and ξ be a transversely oriented codimension-one plane field on M, here 'plane field' means a subbundle of the tangent bundle of M. A smooth function f (resp. vector field X) on M is said to be *admissible* if there is a Riemannian metric g of M so that f (resp. X) is the mean curvature function (resp. mean curvature vector field) of ξ with respect to g. In [6], the author gave a characterization of admissible functions of codimension-one foliations, and in [7], he also gave a characterization of admissible vector fields of codimension-one foliations on M. In this paper, we shall give an extension of the above result on mean curvature vector fields to not necessarily integrable codimension-one plane fields of M. As an application, we have a stability property of mean curvature vector fields, and consequently, that of mean curvature functions, of ξ with respect to variations of codimension-one plane fields.

We shall give some definitions, preliminaries and results in Section 2, and shall prove them in Section 3. Some remarks on the closedness of $B + P\,(X, \xi)$ and on $C_{ad}\,(\xi)$ will be given in Section 4 by using simple foliations on T^2.

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2 Preliminary and result

In this paper, we work in the C^∞-category. In what follows, we always assume that plane fields are of codimension-one and transversely oriented, and that the ambient manifolds are closed, connected, oriented and of dimension $n+1 \geq 2$, unless otherwise stated (see [1], [14] for the generalities on foliations).

Let g be a Riemannian metric of M. Then there is a unique unit vector field orthogonal to ξ whose direction coincides with the given transverse orientation. We denote this vector field by N. Orientations of M and ξ are related as follows: Let $\{V_1, V_2, \ldots, V_n\}$ be an oriented local frame of ξ. Then the orientation of M coincides with the one given by $\{N, V_1, V_2, \ldots, V_n\}$.

We denote the mean curvature of ξ at x with respect to g and N by $h_g(x)$, that is,

$$h_g = \sum_{i=1}^{n} \langle \nabla_{E_i} E_i, N \rangle,$$

where $\langle , \rangle$ means $g(,)$, ∇ is the Riemannian connection of (M, g) and $\{E_1, E_2, \ldots, E_n\}$ is an oriented local orthonormal frame of ξ. Note that this notion is well-defined even if the plane field is not integrable. The vector field $H_g = h_g N$ is called the *mean curvature vector field* of ξ with respect to g. A smooth function f on M is called *admissible* if $f = -h_g$ for some Riemannian metric g (cf. [4], [15]). We also call a vector field X on M *admissible* if $X = H_g$ for some Riemannian metric g. A characterization of admissible functions of codimension-one foliations is given in [6] (see also [4], [5], [15]):

Theorem 1. For any vector field Z transverse to a codimension-one plane field ξ of a closed oriented manifold M, there is a smooth function on M with $\mathrm{supp}(f) = M$ so that fZ is admissible.

We shall recall the characterization of admissible vector fields given in [7].

Define an n-form χ_ξ on M by

$$\chi_\xi(X_1, \ldots, X_n) = \det(\langle E_i, X_j \rangle)_{i,j=1,\ldots,n} \text{ for } X_j \in TM.$$

If the plane field ξ is integrable, then the restriction $\chi_\xi | L$ is the volume element of $(L, L|g)$ for a leaf L of ξ. If the plane field ξ is of codimension-one, then the following well-known formula for foliations holds for not necessarily integrable plane fields.

Proposition R (Rummler [9]). $d\chi_\xi = -h_g dV(M, g) = \mathrm{div}_g(N) dV(M, g)$, where $dV(M, g)$ is the volume element of (M, g) and $\mathrm{div}_g(N)$ is the divergence of N with respect to g, that is, $\mathrm{div}_g(N) = \sum_{i=1}^{n} \langle \nabla_{E_i} N, E_i \rangle$.

Now recall the set-up by Sullivan [12]. Let D_p be the space of p-currents, and D^p be the space of differential p-forms on M with the C^∞-topology. It is well known that D^p is the dual space of D_p (cf. Schwartz [10]). Let $x \in M$ and $\{e_1, \ldots, e_n\}$ be an oriented basis of ξ_x. We define the Dirac current

$\delta_{e1\wedge\cdots\wedge en}$ by

$$\delta_{e1\wedge\cdots\wedge en}(\phi) = \phi_x(e_1 \wedge \cdots \wedge e_n) \text{ for } \phi \in D^n,$$

and set C_ξ to be the closed convex cone in D_n spanned by Dirac currents $\delta_{e1\wedge\cdots\wedge en}$ for all oriented basis $\{e_1, \ldots, e_n\}$ of ξ_x and $x \in M$. We denote a base of C_ξ by $\mathbf{C}_\xi$, which is an inverse image $L^{-1}(1)$ of a suitable continuous linear functional $L : D_n \to \mathbf{R}$. It is known that the base $\mathbf{C}_\xi$ is compact if L is suitably chosen (see Sullivan [12]). In the following, we assume that $\mathbf{C}_\xi$ is compact.

Let X be a vector field on M. Denote by $P(X, \xi)$ the closed linear subspace of D_n generated by all the Dirac currents $\delta_{X(x)\wedge v1\wedge\cdots\wedge vn-1}$ with $v_1, \ldots, v_{n-1} \in \xi_x$ and $x \in M$ (see [11] for more details), where $\delta_{X(x)\wedge v1\wedge\cdots\wedge vn-1}$ is defined by

$$\delta_{X(x)\wedge v1\wedge\cdots\wedge vn-1}(\phi) = \phi_x(X(x) \wedge v_1 \wedge \cdots \wedge v_{n-1}) \text{ for } \phi \in D^n.$$

Let $\partial : D_{n+1} \to D_n$ be the boundary operator and set $B = \partial(D_{n+1})$.

A characterization of admissible vector fields for codimension-one foliations is given in [7] (see also [4], [5], [15]). We extend this result to not necessarily integrable codimension-one plane fields ξ.

Theorem 2. For a vector field X on M, the following two conditions are equivalent.
(1) X is admissible.
(2) There are a volume element dV, a non-vanishing vector field Z transverse to ξ whose direction coincides with the given transverse orientation of ξ, a smooth function f on M, and a neighborhood U of $0 \in D_n$ such that
  (i) $X = -fZ$,
  (ii) $\int_M f dV = 0$,
  (iii) $\int_c f dV = 0$ for all $c \in \partial^{-1}(P(X, \xi) \cap B)$, and
  (iv) $\inf\{\int_c f dV \mid c \in \partial^{-1}((\boldsymbol{C}_\xi + P(X, \xi) + U) \cap B)\} > 0$.

Note that the conditions (ii) and (iv) in this theorem mean that the function f is admissible. In the case when $X \equiv 0$, these conditions become $\boldsymbol{C}_\xi \cap B = \emptyset$, which is equivalent to the 'tautness' of ξ. In Section 4, we shall give an example of one-dimensional foliation on T^2, whose space $B + P(X, \xi)$ is not closed.

As an application, we show a stability property of mean curvature vector fields with respect to perturbations of codimension-one plane fields. When we consider perturbations of plane fields, we consider the topology on the set of C^∞ plane fields by taking the C^∞- topology on the space of sections from M to the oriented Grassmann bundle of all oriented n-planes of the tangent space to M at each point.

Theorem 3. Let X be an admissible vector field of a codimension-one plane field ξ, then X is also admissible for codimension-one plane fields ξ' sufficiently close to ξ.

As a corollary to this theorem, we have the following result.

Corollary. If $f \in C_{ad}(\xi)$ and ξ' is sufficiently close to ξ, then $f \in C_{ad}(\xi')$, where $C_{ad}(\xi)$ is the set of all admissible functions for ξ on M.

3 Proof of Theorems

As Theorems 1 and 2 are proved by the same way as in [7], we only give outlines of the proofs. In order to prove Theorem 1, we need some lemmas.

Lemma 1. Let M be a closed manifold and N be a non-vanishing vector field on M. there is a smooth function φ on M such that supp$N(\varphi) = M$.

Lemma 2. Let M, N and φ be as in Lemma 1. For any smooth function h on M, there is a positive constant $\alpha > 0$ so that supp$(h - \alpha N(\varphi)) = M$.

The following lemma is proved for codimension-one foliations in [3], Lemma 3, where the term in the equality $H' = e^{-2\psi} H$ in (ii) should be corrected by $H' = e^{-\psi} H$. This also holds for non-integrable codimension-one plane fields.

Lemma 3. Let ξ be a codimension-one plane field of a Riemannian manifold (M, g), N be a unit vector filed orthogonal to ξ defined as in Section 2, and h be the mean curvature function of ξ with respect to g.

(i) If $\bar{g} = e^{2\psi} g$, then $\bar{h} = e^{-\psi}(h - N(\psi))$, where $\bar{h}$ is the mean curvature function of ξ with respect to $\bar{g}$ and the unit vector field $\bar{N}$ orthogonal to ξ with respect to $\bar{g}$ defined as in Section 2.

(ii) If $\bar{g}|\xi \otimes TM = g|\xi \otimes TM$ and $\bar{g}(U, V) = e^{2\psi} g(U, V)$ for U and V orthogonal to ξ, then $\bar{h} = e^{-\psi} h$.

(iii) Let $Z = \varphi N + F$ be a vector field on M with $\varphi > 0$ and $F \in \Gamma(\xi)$. Define a Riemannian metric $\bar{g}$ on M as follows: $\bar{g} = g$ on ξ, Z is a unit vector field and orthogonal to ξ with respect to $\bar{g}$. Then we have $\bar{h} = \varphi h + F(\log \varphi) - \mathrm{div}_g(F)$.

We give here another proof of this lemma by presenting a unified form given in [8].

Proposition. Let ξ be a codimension-one plane field of a Riemannian manifold (M, g), N be the unit vector field orthogonal to ξ defined as in Section 2, and h be the mean curvature function of ξ with respect to g. Let $\bar{g}$ be another Riemannian metric of M and $\bar{N}$ be the unit vector field orthogonal to ξ with respect to $\bar{g}$. Set $\bar{N} = \sigma N + F$ for a positive smooth function σ on M and $F \in \Gamma(\xi)$. Further, also set $\bar{\chi}_{\xi}|_{\xi} = \varphi\, \chi_{\xi}|_{\xi}$ for a positive smooth function φ on M. Then, for the mean curvature $\bar{h}$ of ξ with respect to $\bar{g}$, we have

$$\bar{h} = \sigma h - \sigma N(\log \varphi) - F(\log \frac{\varphi}{\sigma}) - \mathrm{div}_g(F).$$

Lemma 3 can be easily derived from this proposition, and Theorem 1 follows directly from these three Lemmas by modifying an arbitrarily given Riemannian metric on M (see [7] for details). To prove Theorem 2, we follow the proof given in [4] with some modifications motivated by [11] (see also Sullivan [12]). To do this we need a Hahn- Banach Theorem of the following form (cf. [2]):

Theorem of Hahn-Banach. Let V be a Fréchet space, W be a closed subspace of V, and C be a compact convex cone at the origin $0 \in V$. Let $\rho : W \to \mathbf{R}$ be a continuous linear functional of W with $\rho(v) > 0$ for $v \in C \cap W \setminus \{0\}$. Then there is a continuous extension $\eta : V \to \mathbf{R}$ of ρ so that $\eta(v) > 0$ for $v \in C \setminus \{0\}$.

Proof of Theorem 2.

(1) ⇒ (2) : Assume that there is a Riemannian metric g of M so that X is the mean curvature vector of ξ. Let N be the unit vector field orthogonal to ξ, and χ_ξ be the n-form defined as in Section 2. If $\mathbf{C}_\xi$ is chosen to be $L^{-1}(1)$ of a continuous linear functional $L : D_n \to \mathbf{R}$ with $\mathbf{C}_\xi$ being compact, as $\chi_\xi : D_n \to \mathbf{R}$ is also continuous, there is a positive constant $\varepsilon > 0$ such that $\chi_\xi \geq \varepsilon > 0$ on $\mathbf{C}_\xi$. We choose $U = \chi_\xi^{-1}(]-\varepsilon/2, \varepsilon/2[)$ as a neighborhood of $0 \in D_n$, where $]a, b[$ is an open interval in $\mathbf{R}$. Set $dV = dV(M, g)$, $Z = N$, and $f = \mathrm{div}_g(N)$. It can be shown that these dV, Z, f, and U satisfy the conditions (i) ~ (iv) in (2). Note that the integrability of ξ is not used in the argument.

(2) ⇒ (1) : Let dV, Z, f, U be as in the conditions of (2). Condition (ii) implies that $fdV = d\phi$ for some $\phi \in D^n$. By the duality of D_p and D^p due to Schwartz, we can regard ϕ as a continuous linear functional $k : D_n \to \mathbf{R}$. By condition (iii), we may assume that $k|(P(X, \xi) \cap B) = 0$. Extend $k : B \to \mathbf{R}$ to $\tilde{k}$ defined on the subspace $P(X, \xi) + B$ by defining $\tilde{k}(z + b) = k(b)$ for $z \in P(X, \xi)$ and $b \in B$. As $k|(P(X, \xi) \cap B) = 0$, this is well-defined and is continuous on $P(X, \xi) + B$. Note that, by condition (iv), $\tilde{k} > 0$ on $C_\xi \cap (P(X, \xi) + B) \setminus \{0\}$. Extend $\tilde{k}$ continuously to κ defined on the closed subspace $W = \overline{P(X, \xi) + B}$. It can be shown that $\kappa(v) > 0$ for $v \in C_\xi \cap W \setminus \{0\}$ (cf. [7]), and the Hahn-Banach Theorem quoted above can be applied to the case $V = D_n$, $W = \overline{P(X, \xi) + B}$, $C = C_\xi$ and $\rho = \kappa$. Thus, we have a continuous linear map $\eta : D_n \to \mathbf{R}$ with $\eta|_B = k|_B$, $\eta(v) > 0$ for $v \in C_\xi \setminus \{0\}$, and $\eta(z) = 0$ for $z \in P(X, \xi)$. By the duality due to Schwartz, we have an n-form χ on M so that $\chi > 0$ on ξ, $d\chi = fdV$, and $\iota_X \chi = 0$, where ι_X is the interior product. Now define a Riemannian metric g as Sullivan did in [13], and deform it as in Lemma 3, we have the desired Riemannian metric.

Remark Note that if the subspace $P(X, \xi) + B$ is closed, it is easy to see that the condition (iv) can be weakened by the following condition, which does not need any existence of U:

$$\int_c fdV > 0 \text{ for all } c \in \partial^{-1}((\mathbf{C} + P(X, \xi)) \cap B).$$

However, in general, the space $P(X, \xi) + B$ is not closed as is seen in Section 4.

Proof of Theorem 3.

Let X be an admissible vector field of ξ. In order to see that X is also admissible for ξ' near ξ, we show that the conditions in Theorem 2 are satisfied for X and ξ'. To do this, we choose a Riemannian metric g on M so that X is the mean curvature vector field of ξ. Let N be the unit vector field orthogonal to ξ, χ_ξ be the n-form defined as in Section 2, and dV be the volume element of (M, g). Note that $X = -fN$ and $d\chi_\xi = fdV$. As the n-form χ_ξ is a continuous liner functional $C_n \to \mathbf{R}$ due to the duality of Schwartz, and $C_\xi \cap \chi_\xi^{-1}(1)$ is easily seen to be compact, we can take $C_\xi \cap \chi_\xi^{-1}(1)$ as the base $\mathbf{C}$ of the cone C_ξ. Take a neighborhood V of the origin as $V = \chi_\xi^{-1}(]-\varepsilon, \varepsilon[)$ for sufficiently small $\varepsilon > 0$.

We take f, $Z = N$, dV and $U = V$ as above and show that the conditions are also satisfied for ξ'. The conditions (i) and (ii) are clearly satisfied by definition. Now we show that $P(X, \xi') = P(X, \xi)$. This clearly implies that the condition (iii) for ξ' is also satisfied. As the generators for $P(X, \xi')$ are of the form $\delta_{X(x) \wedge v1 \wedge \cdots \wedge vn-1}$ with $v_1, \ldots, v_{n-1} \in \xi'_x$, and $v_i = a_i X(x) + e_i$ with $e_i \in \xi_x$ for $i = 1, \ldots, n-1$, it follows that $\delta_{X(x) \wedge v1 \wedge \cdots \wedge vn-1} = \delta_{X(x) \wedge e1 \wedge \cdots \wedge en-1}$ if $X(x) \neq 0$ and $\delta_{X(x) \wedge v1 \wedge \cdots \wedge vn-1} = 0$ if $X(x) = 0$. This shows $P(X, \xi') = P(X, \xi)$.

Finally we show that the condition (iv) is satisfied. Set $\mathbf{C}' = C_{\xi'} \cap \chi_\xi^{-1}(1)$. We show that the set $\mathbf{C}'$ is also compact, thus, is a base of $C_{\xi'}$. To see this, as the set $\mathbf{C}'$ is closed, we need only to show that $\mathbf{C}'$ is bounded. This is done if we can show that the set $\eta(\mathbf{C}') \subset \mathbf{R}$ is bounded for any fixed n-form η on M (cf. Schwartz [10], Sullivan [12]). Let m be the maximum value of η on any unit n-vector (with respect to g) of M. In the following, we denote $\delta_{v1 \wedge \cdots \wedge vn}$ by $v_1 \wedge \cdots \wedge v_n$ for simplicity. For $v_1 \wedge \cdots \wedge v_n \in \mathbf{C}'$, we can choose an orthonormal basis $\{e_1, \ldots, e_n\}$ of ξ_x and real numbers $a_1, \ldots, a_n$ such that $wi = a_i N + e_i \in \mathbf{C}'$ for $i = 1, \ldots, n$ and $v_1 \wedge \cdots \wedge v_n = w_1 \wedge \cdots \wedge w_n$. It follows that

$$v_1 \wedge \cdots \wedge v_n = \sum_{i=1}^{n} (-1)^{i-1} a_i N \wedge e_1 \wedge \cdots \hat{e}_i \cdots \wedge e_n + e_1 \wedge \cdots \wedge e_n,$$

where $\hat{e}_i$ denotes the elimination of ei from $N \wedge e_1 \wedge \cdots \wedge e_n$. As ξ' is close to ξ, we may assume $\sum_{i=1}^{n} = |a_i| < \epsilon$ for sufficiently small $\epsilon > 0$. Thus, we have

$$|\eta(v_1 \wedge \cdots \wedge v_n)| \leq \sum_{i=1}^{n} |a_i| m + m = m(\sum_{i=1}^{n} |a_i| + 1) < (\epsilon + 1)m.$$

Next consider a finite sum $c = \sum a_i \delta_i \in \mathbf{C}'$, where $a_i > 0$ and $\delta_i \in \mathbf{C}_{\xi'}$. As $1 = \chi_\xi(c) = \sum a_i \chi_\xi(\delta_i) = \sum \chi_\xi(\delta_i) a_i \chi_\xi(\delta_i / \chi_\xi(\delta_i))$, we may assume that $\delta_i \in \mathbf{C}'$, $\sum a_i = 1$ and $a_i > 0$. It follows that $|\eta(c)| \leq \sum a_i |\eta(\delta_i)| < \sum a_i (1 + \epsilon) m = (1 + \epsilon) m$. As the set of elements of the form of finite sums are dense in $\mathbf{C}'$, $\eta(\mathbf{C}')$ is bounded. Therefore the set $\mathbf{C}'$ is compact. Now we show that for $V = \chi_\xi^{-1}(]-\varepsilon, \varepsilon[)$ we have

$$\inf\{\int_{c'} fdV \mid c' \in \partial^{-1}((\mathbf{C}' + P(X, \xi') + V) \cap B)\} > 0$$

by assuming that for $U = \chi_\xi^{-1}(]-2\varepsilon, 2\varepsilon[)$

$$\inf\{\int_{c'} fdV \mid c \in \partial^{-1}((\mathbf{C} + P(X, \xi) + U) \cap B)\} > 0.$$

By the above argument, as $\mathbf{C}$ and $\mathbf{C}'$ are contained in $\chi_\xi^{-1}(1)$, it follows that $\mathbf{C}' \subset \mathbf{C} + \chi_\xi^{-1}(0)$. By the definitions of V and U, we have $\mathbf{C}' \subset \mathbf{C} + V$, and, thus, $\mathbf{C}' + V \subset \mathbf{C} + V + V \subset \mathbf{C} + U$. As $P(X, \xi) = P(X, \xi')$, we have $\mathbf{C}' + P(X, \xi') + V \subset \mathbf{C} + P(X, \xi) + U$, which implies $\partial^{-1}((\mathbf{C}' + P(X, \xi') + V) \cap B) \subset \partial^{-1}((\mathbf{C} + P(X, \xi) + U) \cap B)$. This completes the proof.

4 Concluding remark

In this section, we give a simple example which shows that the space $P(X, \xi) + B$ is not closed, and discuss a property of $C_{ad}(\xi)$ related to our Corollary.

Let T^2 be a two dimensional torus with the canonical coordinates (x, y), and $\mathcal{F}$ be a foliation given by $\{S^1 \times \{y\} \mid y \in S^1\}$. We consider the first S^1-factor as the quotient $[0, 2]/\{0 \sim 2\}$. Define a vector field Z on T^2 by $Z = \bar{h}(x)\partial_x + \partial_y$, where $h : [0, 1] \to \mathbf{R}$ is a smooth function satisfying the conditions $h(0) = h(1) = 0$ and $h(x) > 0$ for $x \in]0, 1[$, and $\bar{h}$ is defined by $\bar{h}(x) = h(x)$ for $x \in [0, 1]$ and $\bar{h}(x) = -h(2 - x)$ for $x \in [1, 2]$. We further assume that h is chosen, if we regard Z as a foliation, so that the holonomy groups along the leaves $\{0\} \times S^1$, $\{1\} \times S^1$ are infinitely tangent to the identity maps. Note that Z is invariant under the rotations along the second S^1-factor. Let ϕ_t be the one-parameter group on T^2 generating Z. Consider the closed interval $I = [1/2, 3/2] \times \{0\} \subset S^1 \times S^1$, and set $c_k = \{\phi_t(x) \in S^1 \times S^1 \mid x \in I, -k \le t \le k\}$. Note that $\phi_k(x) \in S^1 \times \{0\}$ for $x \in I$ and $k \in \mathbf{Z}$. It follows that ∂c_k is contained in $S^1 \times \{0\} + P(Z, \mathcal{F})$. As the holonomy of Z is expanding along $\{1\} \times S^1$ and contracting along $\{0\} \times S^1$, it follows that $\partial c_k \to S^1 \times \{0\} = L_0 \in \mathcal{F}$ modulo $P(Z, \mathcal{F})$. By Theorem 1, we can find a smooth function f on T^2 so that $\mathrm{supp}(f) = T^2$, and $X = -fZ$ is admissible. Because $P(-fZ, \mathcal{F}) = P(Z, \mathcal{F})$, we have $L_0 \in \overline{P(X, \mathcal{F}) + B}$. Recall that $B = \partial D_2$. But, as it is clear that $L_0 \notin P(X, \mathcal{F}) + B$, the space $P(X, \mathcal{F}) + B$ is not closed.

According to our Corollary, if $f \in C_{ad}(\xi)$ and m is sufficiently large, then as ξ_m is sufficiently close to ξ with respect to the C^∞-topology of plane fields, it follows that $f \in C_{ad}(\xi_m)$ for sufficiently large m. This seems to imply that $C_{ad}(\xi) \subset C_{ad}(\xi_m)$ for sufficiently large m. But, as m depends on f, this does not hold in general. We give such an example of one-dimensional foliations on T^2. To do this, recall a characterization of admissible functions of codimension-one foliations.

Theorem (Oshikiri [6]). f is admissible for $\mathcal{F}$ if and only if there is a volume form dV on M satisfying the following two conditions:

(1) $\int_M f dV = 0$,

(2) $\int_D f dV > 0$ for every (+)-fcd D, where '(+)-fcd' means a compact saturated domain of M with N being outward everywhere on ∂D.

Let T^2 be the two dimensional torus with the canonical coordinates (x, y), and consider the first S^1-factor as the quotient $[0, 2]/\{0 \sim 2\}$. Set $A = [0, 1] \times S^1$, $B = [1, 2] \times S^1$, and consider Reeb foliations on them. The orientation is given so that A is (+)-fcd. Denote the rotation of angle θ along the first S^1-factor by R_θ, and set $\mathcal{F}_m = (R_{\varepsilon/m})^* \mathcal{F}$ for sufficiently small fixed $\varepsilon > 0$. Then, for sufficiently large m, $\mathcal{F}_m$ is sufficiently close to $\mathcal{F}$ with respect to the C^∞-topology. We show that $C_{ad}(\mathcal{F}) \not\subset C_{ad}(\mathcal{F}_m)$ and $C_{ad}(\mathcal{F}_m) \not\subset C_{ad}(\mathcal{F})$ for all m.

As $\mathrm{Int}(A \backslash (R_{\varepsilon/m})^* A) \neq \emptyset \neq = \mathrm{Int}(B \backslash (R_{\varepsilon/m})^* B)$, we can choose $p, v \in \mathrm{Int}((R_{\varepsilon/m})^* A \cap B)$ and $q, u \in \mathrm{Int}(A \cap (R_{\varepsilon/m})^* B)$. Thus we can find smooth functions f_1, f_2 on T^2 so that $f_1(x) < 0$ near p, $f_1(x) > 0$ near q and $f_1(x) = 0$ elsewhere, and $f_2(x) < 0$ near u, $f_2(x) < 0$ near v and $f_2(x) = 0$ elsewhere. By the characterization, it is clear that $f_1 \in C_{ad}(\mathcal{F})$ but $f_1 \notin C_{ad}(\mathcal{F}_m)$, and that $f_2 \in \mathrm{C}_{\mathrm{ad}}(\mathcal{F})$ but $f_2 \in C_{ad}(\mathcal{F}_m)$.

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