Ann. Rep. Fac. Educ., Iwate Univ., Vol.70 (Mar. 2011), 81 ~ 87

On Curvatures of k-Adapted Metrics to Contact 3-Manifolds

Gen-ichi OSHIKIRI *
(Received Jan. 11, 2011)

Abstract

Though there are many differential geometric results on contact metric structures, adapted metrics for contact structures introduced by S. Chern and R. Hamilton are not studied. In this paper, we study curvatures arising from k-adapted metric to contact structures on 3-manifolds, which is an extended notion of adapted metrics introduced by S. Chern and R. Hamilton, and give some results that also hold for contact metric structures, especially, on K-contact metric structures. We also study the Gaussian curvature of ξ introduced by V. Krouglov.

1 Introduction

Though there are many differential geometric results on contact metric structures (cf. [2]), adapted metrics to contact structures introduced by S. Chern and R. Hamilton [3] are not studied. In this paper, we introduce k-adapted metrics to contact structures on 3-manifolds, where k is a positive constant, and study curvatures arising from adapted metrics. We give some results that also hold for contact metric structures, especially, on K-contact metric structures. We also study the Gaussian curvature of ξ introduced by V. Krouglov [6]. We shall give preliminaries and auxiliary results in §2, and present and prove main results in §3. In §4, we give some examples.

2 Preliminaries and auxiliary results

In this paper, we work in the C^∞-category. In what follows, we always assume that contact structures are given by one-form ω, and that the ambient manifolds are connected, oriented and of dimension 3, unless otherwise stated (see [2], [4], [7] for the generalities on contact structures).

* Faculty of Education, Iwate University

Let ξ = Ker ω, and assume that $\omega \wedge d\omega > 0$ on M. Let g be a Riemannian metric of M. For a positive constant k, a metric g is called *k-adapted* to the contact form ω if there is a unit vector field N orthogonal to ξ such that $\omega(N) = 1$, $\iota_N d\omega = 0$ and $\omega \wedge d\omega = kdV(M, g)$, where $dV(M, g)$ is the volume element of the Riemannian manifold (M, g). In case $k = 2$, g is adapted in the sense of [3]. Thus, if we give an oriented local orthonormal frame $\{N, E, F\}$ with $E, F \in \Gamma(\xi)$, then we have $\omega \wedge d\omega = kdV(M, g) = kN^* \wedge E^* \wedge F^*$. By definition, $d\omega/k$ is the volume element along ξ. It follows that ξ is minimal in the sense that

$$\langle \nabla_E N, E \rangle + \langle \nabla_F N, F \rangle = 0,$$

where ∇ is the Riemannian connection of (M, g).

As $\iota_N d\omega = 0$, for any section V of ξ we have

$$0 = \iota_N d\omega(V) = d\omega(N, V) = -\omega([N, V]) = \langle \nabla_N N, V \rangle .$$

Thus we have

Lemma 1. $\nabla_N N = 0$, that is, the orbits of N are geodesics.

This is known for contact metric structures, which needs stronger conditions than adapted ones (cf. [2]). By calculating $d\omega(E, F) = k$ for an oriented orthonormal basis $\{E, F\}$ of ξ, we have

$$k = d\omega(E, F) = -\langle \nabla_E F - \nabla_F E, N \rangle .$$

It follows that

Lemma 2. For an oriented orthonormal frame $\{N, E, F\}$, we have

$$\langle \nabla_F E, N \rangle = k + \langle \nabla_E F, N \rangle .$$

Define the second fundamental form B of ξ by

$$B(V, W) = \frac{1}{2} \langle \nabla_V W + \nabla_W V, N \rangle$$

for all sections V, W of ξ (cf. [6], [8]). This is a symmetric bilinear form on ξ. As ξ is minimal, it follows that Tr$B = 0$. Define also the *extrinsic curvature* $K_e(\xi)$ by

$$K_e(\xi) = \frac{B(V, V)B(W, W) - B(V, W)^2}{\langle V, V \rangle \langle W, W \rangle - \langle V, W \rangle^2},$$

where V, W are two linearly independent sections of ξ, and the *Gaussian curvature* $K_G(\xi)$ by $K_G(\xi) = K(\xi) + K_e(\xi)$ (cf. [6]). Note that $K_e(\xi) \leq 0$ because Tr$B = B(E, E) + B(F, F) = 0$ for an oriented orthonormal basis $\{E, F\}$ of ξ. Let $\{N, E, F\}$ be an oriented orthonormal frame of M. Now calculate $K(\xi)$.

$$\begin{aligned}
K(\xi) = K(E, F) &= \langle R(E, F)F, E \rangle \\
&= \langle \nabla_E \nabla_F F, E \rangle - \langle \nabla_F \nabla_E F, E \rangle - \langle \nabla_{[E, F]} F, E \rangle \\
&= E\langle \nabla_F F, E \rangle + F\langle \nabla_E E, F \rangle - \langle \nabla_E E, F \rangle^2 - \langle \nabla_F F, E \rangle^2 \\
&\quad - \langle \nabla_E E, N \rangle \langle \nabla_F F, N \rangle + \langle \nabla_E F, N \rangle \langle \nabla_F E, N \rangle + k \langle \nabla_N F, E \rangle .
\end{aligned}$$

As $\mathrm{div}(E) = \langle \nabla_E E, E\rangle + \langle \nabla_F E, F\rangle + \langle \nabla_N E, N\rangle = -\langle \nabla_F F, E\rangle$, $\mathrm{div}(F) = -\langle \nabla_E E, F\rangle$ and $-\mathrm{div}(\mathrm{div}(E)E + \mathrm{div}(F)F) = \mathrm{div}(\nabla_E E + \nabla_F F)$, we have

Lemma 3. For an oriented orthonormal frame $\{N, E, F\}$ of M, we have

$$K(\xi) = \mathrm{div}(\nabla_E E + \nabla_F F) - \langle \nabla_E E, N\rangle\langle \nabla_F F, N\rangle + \langle \nabla_E F, N\rangle\langle \nabla_F E, N\rangle + k\langle \nabla_N F, E\rangle.$$

Note that the vector field $\nabla_E E + \nabla_F F$ is independent of the choices of local orthonormal frame $\{E, F\}$ of ξ. By using Lemma 1, we have

$$\begin{aligned} K(E, N) &= \langle R(E, N)N, E\rangle \\ &= \langle \nabla_E \nabla_N N, E\rangle - \langle \nabla_N \nabla_E N, E\rangle - \langle \nabla_{[E,N]} N, E\rangle \\ &= -N\langle \nabla_E N, E\rangle + \langle \nabla_E N, \nabla_N E\rangle - \langle [E, N], N\rangle \langle \nabla_N N, E\rangle \\ &\quad - \langle [E, N], E\rangle \langle \nabla_E N, E\rangle - \langle [E, N], F\rangle \langle \nabla_F N, E\rangle. \end{aligned}$$

It follows that

Lemma 4. For an oriented orthonormal frame $\{N, E, F\}$ of M, we have

$$K(E, N) = N\langle \nabla_E E, N\rangle - \langle \nabla_E E, N\rangle^2 - \langle \nabla_E F, N\rangle\langle \nabla_F E, N\rangle - \langle \nabla_N E, F\rangle(\langle \nabla_E F, N\rangle + \langle \nabla_F E, N\rangle).$$

As the Ricci curvature in the direction of N is given by $Ric(N, N) = K(E, N) + K(F, N)$ and $\mathrm{Tr}B = 0$, by Lemma 4, we have

Lemma 5. For an oriented orthonormal frame $\{N, E, F\}$ of M, we have

$$Ric(N, N) = -\langle \nabla_E E, N\rangle^2 - \langle \nabla_F F, N\rangle^2 - 2\langle \nabla_E F, N\rangle\langle \nabla_F E, N\rangle.$$

3 Main results

Firstly, we locally represent covariant derivatives explicitly by using the results in § 2. Let U be an open subset of M.

Proposition 1. Let $\{N, E, F\}$ be an oriented orthonormal frame on U. As $\nabla_N E \perp E, N$, we can set $\nabla_N E = fF$ for a smooth function on U. Set also $\langle \nabla_E E, N\rangle = \lambda$ and $\langle \nabla_E F, N\rangle = \alpha$. Then we have

(1) $\nabla_N N = 0$, $\nabla_N E = fF$, $\nabla_N F = -fE$.

(2) $\nabla_E N = -\lambda E - \alpha F$, $\nabla_E E = -\mathrm{div}(F)F + \lambda N$, $\nabla_E F = \mathrm{div}(F)E + \alpha N$.

(3) $\nabla_F N = -(k+\alpha)E + \lambda F$, $\nabla_F E = \mathrm{div}(E)F + (k+\alpha)N$, $\nabla_F F = -\mathrm{div}(E)E - \lambda N$.

Theorem 1. Under the assumptions in Proposition 1, we get

$$K_G(\xi) = K(\xi) + K_e(\xi) = \mathrm{div}\,(\,\nabla_E E + \nabla_F F\,) - kf - \frac{k^2}{4}.$$

Note that the function $f = \langle \nabla_N E, F \rangle$ can vary as we see in the examples given in §4.As a corollary to this theorem, we have the following by letting k be sufficiently large (see [6] for more general result).

Corollary. If M is closed, then there is a k-adapted metric so that $K_G(\xi) < 0$.

Next, we give an upper bound of the Ricci curvature in the direction of N (cf. [2]).

Theorem 2. $Ric\,(N, N) \leq \frac{k^2}{2}$, where the equality holds if and only if $B = 0$ at the point.

(*Proof.*) Set $t = \langle \nabla_E F, N \rangle$. Then, by Lemma 2, $\langle \nabla_F E, N \rangle = k + t$. By Lemma 5, it follows that

$$Ric\,(N, N) \leq -2t\,(k+t) \leq -2\left(t + \frac{k}{2}\right)^2 + \frac{k^2}{2} \leq \frac{k^2}{2}.$$

If $Ric\,(N, N) = k^2/2$, then $\langle \nabla_E E, N \rangle^2 = \langle \nabla_F F, N \rangle^2 = 0$ and $t = -k/2$. This implies that $B\,(E, E) = B\,(F, F) = 0$ and $2B\,(E, F) = \langle \nabla_E F + \nabla_F E, N \rangle = -k/2 + k/2 = 0$. Thus we have $B = 0$. Conversely, assume that $B = 0$. Then, we have $0 = B\,(E, E) = \langle \nabla_E E, N \rangle$, $0 = B\,(F, F) = \langle \nabla_F F, N \rangle$ and $0 = 2B\,(E, F) = \langle \nabla_E F + \nabla_F E, N \rangle$. By Lemma 2, $\langle \nabla_F E, N \rangle = k + \langle \nabla_E F, N \rangle$. It follows that $\langle \nabla_E F, N \rangle = -k/2$, which shows that $Ric\,(N, N) = k^2/2$.

It seems to be of some interest to see a conclusion of $B = 0$. We investigate a relation between N and B on an open set $U \subset M$.

Theorem 3. On an open set $U \subset M$, $B = 0$ if only if N is a Killing vector field.

(*Proof.*) Note that N is a Killing vector field on U if and only if

$$\langle \nabla_X N, Y \rangle + \langle \nabla_Y N, X \rangle = 0$$

for any vectors X and Y on U. As $\nabla_N N = 0$ by Lemma 1, this is equivalent to the following conditions:

$$\langle \nabla_E N, E \rangle = 0, \; \langle \nabla_F N, F \rangle = 0 \text{ and } \langle \nabla_E N, F \rangle + \langle \nabla_F N, E \rangle = 0.$$

And these conditions can be read as

$$B\,(E, E) = 0, \; B\,(F, F) = 0 \text{ and } B\,(E, F) = B\,(F, E) = 0.$$

This means that $B = 0$ on U, and this completes the proof.

A contact metric structure with N being a Killing vector field is called a K-contact metric structure, and the following result is known for K-contact metric structures (cf. [1], [5]). We can get the same result for our k-adapted metrics with weaker conditions than K-contact metric structures.

Theorem 4. If N is a Killing vector field on M, then the sectional curvatures for planes containing N are always equal to $k^2/4$ at any point of M.

(*Proof.*) By Theorem 2, $B = 0$. Thus, for an oriented orthonormal frame $\{N, E, F\}$ of M, we have

$$\langle \nabla_E N, E \rangle = 0, \langle \nabla_F N, F \rangle = 0 \text{ and } \langle \nabla_E N, F \rangle + \langle \nabla_F N, E \rangle = 0.$$

It is easy to see that $K(N, E) = k^2/4$ by Lemma 4.

Combining these results, we have the following.

Theorem 5. For a k-adapted metric of a contact structure on M, the following conditions are equivalent on an open set $U \subset M$:

(1) $Ric(N, N) = k^2/2$ at any point of U.

(2) $B = 0$ at any point of U.

(3) N is a Killing vector field on U.

Further, in these cases, the sectional curvatures for planes containing N are always equal to $k^2/4$ at any point of U.

We locally represent covariant derivatives explicitly as in Proposition 1. Note that in this case we have $\alpha = -k/2$ and $\lambda = 0$.

Proposition 2. Under the assumptions in Theorem 5 and Proposition 1, we have

(1) $\nabla_N N = 0$, $\nabla_N E = fF$, $\nabla_N F = -fE$.

(2) $\nabla_E N = k/2F$, $\nabla_E E = -\operatorname{div}(F)F$, $\nabla_E F = \operatorname{div}(F)E - k/2N$.

(3) $\nabla_F N = -k/2E$, $\nabla_F E = \operatorname{div}(E)F + k/2N$, $\nabla_F F = -\operatorname{div}(E)E$.

(4) $Ric(N, N) = k^2/2$.

(5) $B = 0$, $K_e(\xi) = 0$.

(6) $K(\xi) = K_G(\xi) = \operatorname{div}(\nabla_E E + \nabla_F F) - kf - k^2/4$.

4. Examples

The first example is due to [7]. Let (R^3, g_0) be the 3-dimensional Euclidean space with the canonical coordinate (x, y, z). Define $\omega = \sin z dx + \cos z dy$. Then, it follows that

$$d\omega = \cos z dz \wedge dx - \sin z dz \wedge dy, \text{ and } \omega \wedge d\omega = dx \wedge dy \wedge dz = dV(\mathrm{R}^3, g_0).$$

Thus, g_0 is a 1-adapted metric to this contact form ω. Define $N = \sin z\partial_x + \cos z\partial_y$, $E = \partial_z$ and $F = \cos z\,\partial_x - \sin z\,\partial_y$. Then, it is easy to see that $\{N, E, F\}$ is an oriented orthonormal frame on R^3. We get covariant derivatives

$$\nabla_N N = 0,\ \nabla_N E = 0,\ \nabla_N F = 0,$$
$$\nabla_E N = \cos z\partial_x - \sin z\partial_y,\ \nabla_E E = 0,\ \nabla_E F = -\sin z\partial N_x - \cos z\partial_y,$$
$$\nabla_F N = 0,\ \nabla_F E = 0,\ \nabla_F F = 0.$$

Thus, we have

$$f = \langle \nabla_N E, F \rangle = 0,\ \lambda = \langle \nabla_E E, N \rangle = 0,\ \alpha = \langle \nabla_E F, N \rangle = -1,$$

and

$$B(E,E) = B(F,F) = 0,\ B(E,F) = B(F,E) = -1/2,$$
$$K(\xi) = 0,\ K_e(\xi) = -1/4,\quad K_G(\xi) = -1/4.$$

The second one is due to [3]. Let (S^3, g_1) be the unit sphere in the 4-dimensional Euclidean space R^4 with the canonical coordinate (x, y, z, w). Define a 1-form ω by

$$\omega = xdy - ydx + zdw - wdz.$$

It is easy to see that $\omega \wedge d\omega = 2dV(S^3, g_1)$. Thus g_1 is a 2-adapted to this contact form ω.
Set $N = x\partial_y - y\partial_x + z\partial_w - w\partial_z$, $E = x\partial_z - z\partial_x - y\partial_w + w\partial_y$ and $F = x\partial_w - w\partial_x - y\partial_z - z\partial_y$.
Then $\{N, E, F\}$ is an oriented orthonormal frame of S^3 with $\omega(N) = 1$ and $E, F \in \operatorname{Ker}\omega$.
We get covariant derivatives

$$\nabla_N N = 0,\ \nabla_N E = -F,\ \nabla_N F = E,$$
$$\nabla_E E = 0,\ \nabla_E F = -N,\ \nabla_E N = F,$$
$$\nabla_F E = N,\ \nabla_F F = 0,\ \nabla_F N = -E.$$

Thus, we have

$$f = \langle \nabla_N E, F \rangle = -1,\ \lambda = \langle \nabla_E E, N \rangle = 0,\ \alpha = \langle \nabla_E F, N \rangle = -1,$$

and

$$B(E,E) = B(F,F) = B(E,F) = B(F,E) = 0,$$
$$K_e(\xi) = 0,\ K_G(\xi) = K(\xi) = 0 - 2(-1) - 2^2/4 = 1.$$

References

[1] D. Blair 1977, Two remarks on contact metric structures, *Tohoku Math. Journ*. 29, 319–324.
[2] D. Blair 2002, *Riemannian geometry of contact and symplectic manifolds*, Progress in Math. 203, Birkhäuser, Boston.
[3] S. Chern and R. Hamilton 1985, On Riemannian metrics adapted to Three-dimensional contact manifolds, *Lecture Notes in Mathematics* 1111, Springer, Berlin, 279–308.
[4] J. Etnyre 2003, Introductory lectures on contact geometry, *Proc. Symp. Pure Math*. 71, 81–107.
[5] Y. Hatakeyama, Y. Ogawa and S. Tanno 1963, Some properties of manifolds with contact metric structure, *Tohoku Math. Journ*. 15, 42–48.
[6] V. Krouglov 2008, The curvature of contact structures on 3-manifolds, *Alg. and Geom. Topology* 8, 1567–1579.
[7] Z. Olszak 1979, On contact metric manifolds, T*ohoku Math. Journ.* 31, 247–253.
[8] B. Reinhart 1977, The second fundamental form of a plane field, *Journ. of Diff. Geom.* 12, 619–627.