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The 300th-and-Something Proof of the Pythagorean Theorem — A Combinatorial Equivalent — Dedicated to Professor H. Komiyama on the Occasion of His Retirement

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Abstract

More than 300 different proofs are known for the Pythagorean theorem. Inspired by E. Landau's "unusual proof" for the theorem in trigonometric form, a proof based on the theory of power series is given. Also, a combinatorial equivalent with unexpected appearance is proposed for this most fundamental of all geometric theorems.

1. Introduction

According to some books on ancient Greek mathematics, it is said that the distinction between even numbers and odd numbers dates to the Pythagoras school. In this note, by giving a proof of the Pythagorean theorem, we remark that a very simple "parity" nature of integers is closely related to the Pythagorean theorem.

Many different proofs exist for the Pythagorean theorem. A chapter in the recently published book by E. Maor [3] is entitled "371 Proofs, and Then Some". In addition, there are many web pages on various proofs of the theorem (e.g. [5]).

Here, we treat a proof via trigonometric functions. As is mentioned in Loomis [1], it is said that there are no trigonometric proofs, because all the fundamental formulas of trigonometry are themselves based upon the truth of the Pythagorean theorem in the form

$$\cos^2 x + \sin^2 x = 1 . \quad (*)$$

The difficulty occurs in the process that causes a vicious circle; that is to say, proofs via trigonometric functions should be those to prove the formula (*) itself directly.

E. Landau's proof in this context is introduced as "A Most Unusual Proof" in Maor's above-mentioned

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book ([3],Sidebar 6). The proof is as follows: After defining cosine function and sine function by convergent power series there come several theorems establishing the addition formulas and the even and odd properties for these functions. Then (*) is shown:

$$1 = \cos(x - x) = \cos x \cos(-x) - \sin x \sin(-x) = \cos^2 x + \sin^2 x \,.$$

Regarding Landau's style to avoid relying on geometric intuition, see also an article titled "The Master Rigorist" in Maor [2].

Pushing forward Landau's approach, we arrive at an alternative *unusual* proof that gives an equivalent of the theorem with a simple parity nature of binomial coefficients:

Theorem *The condition that*

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} = 0$$

holds for every positive integer n is equivalent to the Pythagorean theorem.

2. Proof or an exposition of the trick

Since no specific properties of sine and cosine functions are used, we use notation $C(x)$ and $S(x)$ instead of the usual ones: let these functions be defined by convergent power series:

$$C(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

and

$$S(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

respectively.

From Cauchy's rule for the multiplication of two power series (see e.g. [4],1.65), one has

$$C(x)^2 = \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \right)^2 = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} \frac{(-1)^{n-k}}{(2n-2k)!} \right) x^{2n}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left\{\sum_{k=0}^{n}\binom{2n}{2k}\right\}x^{2n}=1+\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}\left\{1+\sum_{k=0}^{n-1}\binom{2n}{2k}\right\}x^{2n}.$$

Similarly

$$S(x)^2=\left(\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right)^2=\sum_{n=0}^{\infty}\left((-1)^n\sum_{k=0}^{n}\frac{1}{(2k+1)!(2n-2k+1)!}\right)x^{2n+2}$$

holds. Replacing $n+1$ with n , one has

$$=\sum_{n=1}^{\infty}\left(-\sum_{k=0}^{n-1}\frac{(-1)^n}{(2k+1)!(2n-(2k+1))!}\right)x^{2n}=\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}\left\{-\sum_{k=0}^{n-1}\binom{2n}{2k+1}\right\}x^{2n}.$$

These we combine to give

$$C(x)^2+S(x)^2=1+\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}\left[1+\sum_{k=0}^{n-1}\left\{\binom{2n}{2k}-\binom{2n}{2k+1}\right\}\right]x^{2n}.$$

Since the inner sum

$$1+\sum_{k=0}^{n-1}\left\{\binom{2n}{2k}-\binom{2n}{2k+1}\right\},$$

the coefficients of the term with degree $2n$ of the power series, is equal to

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k},$$

the formula $C(x)^2+S(x)^2=1$ is obtained.

The inverse implication follows from the uniqueness for coefficients of the convergent power series.

3. Remarks

(1) In the same way, the addition formulas used in Landau's proof can also be derived without any use of differential: the hidden subject of the paper might be an illustration for the well-known reasoning

about the "countability" of the models for our mathematical theories.

(2) The condition $\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}=0$ is nothing but the binomial expansion of $(1-1)^{2n}=0$, which reflects the basic fact that the set of positive integers up to $2n$ has a partition with two parts, even and odd, of same number.

References

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[2] Maor, E., *Trigonometric Delights.* Princeton, New Jersey. Princeton University Press, 2002.

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[5] Web site: *Pythagorean Theorem and its many proofs.* http://www.cut-the-knot.org/pythagoras/index.shtml, (2007)