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Hamiltonian Property of Claw-free Graphs from the View Point of Topological Combinatorics

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Abstract

Let G be a connected claw-free simple graph with $|G| \geq 3$. We show that if the 2-dimensional simplicial complex $\triangle(G)$ associated to G is simply connected then G is hamiltonian. A graph G is said to be $\triangle^1$-connected if every pair of edges are connected by some chain consisting of edges and triangles. We also show that if G is $\triangle^1$-connected then G is hamiltonian.

1 Introduction.

Recently, a new homotopy theoretical approach has been used to study graphs by many authors such as X. Kramer and R. Laudenbacher [9], H. Barcel and X. Kramer [4], and E. Babson et al. [3]. These approaches have been originated from Atkin's papers [1], [2]. On the other hand, as is well-known, L. Lovász used a homotopy theoretical method to solve the Kneser Conjecture [10], and this area has also been studied extensively (cf. [5], [6], [8], [11]). In this paper, we try to use this approach, so called "Topological Combinatorics", to find hamiltonian cycles of claw-free graphs. Note that the complete bipartite graph $K_{1,3}$ is called a claw, and that a graph is called claw-free if it does not contain any claws as induced subgraphs. A graph is called hamiltonian if it contains a cycle passing through all vertices of the graph (see [7] for the fundamentals on graphs).

Let G be a connected simple graph with $|G| \geq 3$. Attach a triangle (i.e., a 2-simplex) to every triangle of G and denote the resulting 2-dimensional simplicial complex by $\triangle(G)$. We show the following.

Theorem 1 If G is a 2-connected claw-free graph with $\triangle(G)$ being simply-connected, then G is hamiltonian.

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Two edges α and ω are said to be 1-*connected* if there is an alternating sequence of edges σ_i and triangles $\triangle_j$ of G

$$P: \alpha = \sigma_0, \triangle_1, \sigma_1, \triangle_2, \sigma_2, \triangle_3, \cdots, \triangle_k, \sigma_k = \omega$$

such that each triangle $\triangle_i$ contains σ_{i-1} and σ_i as edges. Such a sequence will be called a $\triangle$ -*chain*. We denote by $|P|$ the length of the chain P , that is, $|P| = k$ in the above case. A graph G is said to be $\triangle^1$-*connected* if every pair of edges are connected by some $\triangle$ -chain. Note that if G is $\triangle^1$-connected, then G has no cut-vertices and bridges, thus G is 2-connected and 2-edge connected. Though every edge is contained in a triangle if G is $\triangle^1$-connected, G is not necessarily locally-connected as the following example shows: $G = C_6^2 \setminus \{e\}$, where C_6 is a 6-cycle, C_6^2 is the square of C_6, and e is an edge of $C_6^2 \setminus C_6$. Also, a $\triangle^1$-connected graph may contain some claws: consider three triangles and glue them along one edge, which produces a $\triangle^1$-connected non-hamiltonian graph with claws. We also show the following.

Theorem 2 Let G be a $\triangle^1$-connected claw-free graph with $|G| \geq 3$, then G is hamiltonian.

2 Proofs of Theorems.

Let G be a connected simple graph with $|G| \geq 3$. We shall use the notations defined in §1.

(*Proof of Theorem 1.*) We shall show that G is locally-connected. Then by the following theorem of D. Oberly and D. Sumner [12], we have the proof of Theorem 1.

Theorem OS A connected, locally connected claw-free graph of $|G| \geq 3$ is hamiltonian.

Let $v \in V(G)$ be a vertex of G. To show that G is locally connected, by definition, it is sufficient to show that the neighborhood $N(v)$ is connected. Take two vertices $u, w \in N(v)$. Then $vu, vw \in E(G)$. As G is 2-connected, there is a path P in G from u to w, which does not contain v. Thus we have a closed circuit $P \cup \{uv, vw\}$. As $\triangle(G)$ is simply connected, there is a continuous map $f: D^2 \rightarrow \triangle(G)$ with $f(S^1) \subset P \cup \{uv, vw\}$, where D^2 is the 2-dimensional disk and $S^1 = \partial D^2$. We take a $p \in S^1$ so that $f(p) = v$ and denote it by p_0. Considering D^2 as a 2-dimensional simplicial complex, and applying the theorem on simplicial approximation, we can regard f itself a simplicial map. Take the connected component C_0 of $f^{-1}(v)$ that contains the vertex p_0. Collect all simplexes in D^2 which intersects with C_0, and denote it by N_0. As $\partial f(N_0)$ is contained in $\triangle(< N(v) \cup v >)$, and $\partial f(N_0)$ contains uv and wv, we have a trail between u and w in $N(v)$. This shows that G is locally connected, and this completes the proof of Theorem 1.

(*Proof of Theorem 2.*) We shall prove the theorem by reductio ad absurdum. Let $G = (V, E)$ be a $\triangle^1$ -connected claw-free simple graph with $|V| \geq 3$, and C be one of the longest cycle/of G. If $V(C) = V$ then there is nothing to prove, we assume $V \setminus V(C) \neq \emptyset$. Then there are $u \in V(C)$ and $v \notin V(C)$ with $uv \in E$. Orient C, and for $u \in V(C)$ denote by $u+$ (u^-) the successor (the predecessor) of u on C under the orientation of C. Note that if $u \in V(C)$ and $v \notin V(C)$ with $uv \in E$, then $vu^{\pm} \in E$, and as G is claw-free, $u^-u^+ \in E$. We set $\alpha = uv$ and $\omega = uu^-$. As the argument given below is also valid if we replace $\omega = uu^-$ by $\omega = uu^+$, we consider only the case $\omega = uu^-$. As G is $\triangle^1$-connected, there is a chain P connecting α and ω. Among $u \in V(C)$ and $v \notin V(C)$ with $uv \in E$, and P connecting α and ω, we assume that P is the shortest one.

$$P: \alpha = \sigma_0, \triangle_1, \sigma_1, \triangle_2, \sigma_2, \triangle_3, \cdots, \triangle_k, \sigma_k = \omega$$

Thus we can assume $\sigma_i \neq \sigma_j$ and $\triangle_i \neq \triangle_j$ for $i \neq j$. We denote by $\sigma_i P$ the part $\sigma_i, \triangle_{i+1}, \cdots, \triangle_k, \sigma_k = \omega$ of P. We also use the notations $\triangle_i P$, $P\sigma_i$ and $P\triangle_i$ to denote the similar chains.

Now we have to consider several cases according as the vertices in $\triangle_i$'s are in $V(C)$ or not. Firstly, note that if $|P| = 1$ then $\triangle_1 = \{u, v, u^-\}$ and the new cycle $u^+Cu^-vuu^+$ is longer than C, which contradicts the maximality of the length of C. Here we use the notation xCy for $x, y \in V(C)$ that means the part of C beginning at x and ending at y along the given orientation of C.

Assume that $|P| \geq 2$ and let $\triangle_1 = \{u, v, w\}$. We may also assume that $w \neq u^\pm$.

<u>Case 1</u> $w \in V(C)$

In this case, as G is claw-free, $w^-w^+ \in E$.

<u>Case 1.1</u> $\sigma_1 = uw$

Let $\triangle_2 = \{u, w, x\}$. We may assume that $x \neq u^\pm, w^\pm$, for if x is one of them, we can find a cycle longer than C, which contradicts the maximality of C: e.g., if $x = w^-$, then $uw^- \in E$, and $w^-uvwCu^-u^+Cw^-$ is a cycle longer than C.

If $x \notin V(C)$ and $\sigma_2 = ux$, then the chain $ux, \triangle_2 P$ would be a shorter chain than P, which contradicts the minimality of P. Thus, if $x \notin V(C)$, we may also assume that $\sigma_2 = wx$. But, in this case, we can also find a shorter chain $ux, \triangle_2 P$, a contradiction. Therefore, $x \in V(C)$.

If $ux^- \in E$ or $ux^+ \in E$, then we would have a longer cycle, a contradiction: e.g., if $ux^- \in E$, then $x^-uvwxCu^-u^+Cw^-w^+Cx^-$ is a longer cycle than C. Thus, as G is claw-free and $ux^\pm \notin E$, it follows that $x^-x^+ \in E$. If $u^+x \in E$, then we would have a longer cycle $xu^+Cw^-w^+Cx^-x^+Cuvwx$, a contradiction. Thus, $u^+x \notin E$. As it is easy to see that $vu^+ \notin E$, it follows that $vx \in E$ because G is claw-free. If $\sigma_2 = ux$, then P can be shorten to $\sigma_0, \{u, v, x\}, \sigma_2 P$, a contradiction. We may assume that $\sigma_2 = wx$, and let $\triangle_3 = \{w, x, y\}$.

<u>Case 1.1.a</u> $y \notin V(C)$

It is easy to see that $ux^+ \notin E$ and $yx^+ \notin E$. As G is claw-free, $uy \in E$. If $\sigma_3 = wy$, then

$$\sigma_0, \triangle_1, \sigma_1 = uw, \{u, w, y\}, \sigma_3 = wy, \triangle_4 P$$

is a shorter chain than P, a contradiction. If $\sigma_3 = xy$, then

$$\sigma_0, \{u, v, x\}, ux, \{u, x, y\}, \sigma_3 = xy, \triangle_4 P$$

is a shorter chain than P, a contradiction.

<u>Case 1.1.b</u> $y \in V(C)$

As the triangles in P are mutually different, it follows that $u \neq y$. If $y = u^-$, then there is a cycle $y = u^-Cw^-w+Cx^-x^+Cuvwxy$ longer than C, a contradiction. Similarly, if $y = u^+$, then there is a cycle longer than C. Thus $y \neq u, u^\pm$.

If $wy^+ \in E$, then there is a cycle $y^+Cu^- u^+Cw^- w^+Cx^- x^+Cyxvuwy^+$ longer than C, a contradiction. Similarly, if $wy^- \in E$, then there is a cycle longer than C. Thus, $wy^{\pm} \notin E$. As G is claw-free, it follows that $y^- y^+ \in E$. If $ux^- \in E$, then there is a cycle $x^+Cu^- u^+Cw^- w^+Cx^- uvwxx^+$ longer than C, a contradiction. Thus $ux^- \in E$. Similarly, if $yx^- \in E$, then there is a cycle longer than C. Thus, $yx^- \notin E$. As G is claw-free, it follows that $uy \in E$. Therefore, if $\sigma_3 = wy$, then there is a shorter chain σ_0, $\triangle_1$, σ_1, $\{u, w, y\}$, $\sigma_3 = wy$, $\triangle_4 P$ than P , a contradiction. If $\sigma_3 = xy$, then there is a shorter chain σ_0, $\{u, v, x\}$, ux, $\{u, x, y\}$, $\sigma_3 = xy$, $\triangle_4 P$ than P , a contradiction.

Case 1.2 $\sigma_1 = vw$

Let $\triangle_2 = \{v, w, x\}$.

Case 1.2.a $x \notin V(C)$

If $uw^- \in E$, then there is a cycle $w^- uvxwCu^- u^+Cw^-$ longer than C, a contradiction. Similarly, if $xw^- \in E$, then there is a cycle longer than C. Thus, uw^-, $xw^- \notin E$. As G is claw-free, it follows that $ux \in E$. If $\sigma_2 = vx$, then there is a shorter chain σ_0, $\{u, v, x\}$, $\sigma_2 P$ than P , a contradiction. Thus we may assume that $\sigma_2 = wx$. Let $\triangle_3 = \{w, x, y\}$.

Case 1.2.a.(i) $y \in V(C)$

As $xy^{\pm} \notin E$ and G is claw-free, it follows that $y^- y^+ \in E$. If $vw^- \in E$, then there is a cycle $w^- vxywCy^- y^+Cw^-$ longer than C, a contradiction. Similarly, if $yw^- \in E$, then there is a cycle longer than C. Thus, vw^-, $yw^- \notin E$. As G is claw-free, it follows that $vy \in E$. If $\sigma_3 = xy$, then there is a shorter chain σ_0, $\{u, v, x\}$, vx, $\{v, x, y\}$, $\sigma_3 P$ than P , a contradiction. If $\sigma_3 = wy$, then there is a shorter chain σ_0, $\triangle_1$, σ_1, $\{v, w, y\}$, $\sigma_3 P$ than P , a contradiction.

Case 1.2.a.(ii) $y \notin V(C)$

As vw^-, $yw^- \notin E$ and G is claw-free, it follows that $vy \in E$. If $uw^- \in E$, then there is a cycle $w^- uvxywCu^- u^+Cw^-$ longer than C, a contradiction. Similarly, if $yw^- \in E$, then there is a cycle longer than C. Thus, uw^-, $yw^- \notin E$. As G is claw-free, it follows that $uy \in E$. If $\sigma_3 = xy$, then there is a shorter chain σ_0, $\{u, v, x\}$, vx, $\{v, x, y\}$, $\sigma_3 P$ than P , a contradiction. If $\sigma_3 = wy$, then there is a shorter chain σ_0, $\triangle_1$, σ_1, $\{v, w, y\}$, $\sigma_3 P$ than P , a contradiction.

Case 1.2.b $x \in V(C)$

We may assume that $x \neq u, u^{\pm}$. As $vx^{\pm} \notin E$ and G is claw-free, we have $x^- x^+ \in E$.

Case 1.2.b.(i) $\sigma_2 = vx$

Let $\triangle_2 = \{v, x, y\}$.

Assume $y \notin V(C)$. As yx^-, $wx^- \notin E$ and G is claw-free, we have $yw \in E$. As yw^-, $uw^- \notin E$ and G is claw-free, we have $uy \in E$. Thus, if $\sigma_3 = vy$, then there is a shorter chain σ_0, $\{u, v, y\}$, $vy = \sigma_3 P$ than P , a contradiction. If $\sigma_3 = xy$, then there is a shorter chain σ_0, $\{u, v, y\}$, vy, $\triangle_2$, $\sigma_3 P$ than P , a contradiction.

Assume $y \in V(C)$. It is easy to see that y $\neq u, u^{\pm}, w$. As $vy^{\pm} \notin E$ and G is claw-free, we have $y^- y^+ \in E$. Also, as yw^-, $uw^- \notin E$, yx^-, $wx^- \notin E$, and G is claw-free, we have uy, $yw \in E$. Thus, if $\sigma_3 = vy$, then there is a shorter chain σ_0, $\{u, v, y\}$, $vy = \sigma_3 P$ than P , a contradiction. If $\sigma_3 = xy$, then there is a shorter chain σ_0, $\{u, v, y\}$, vy, $\{v, x, y\}$, $xy = \sigma_3 P$ than P , a contradiction.

Case 1.2.b.(ii) $\sigma_2 = wx$

Let $\triangle_2 = \{w, x, y\}$.

Assume $y \in V(C)$. As $xy^{\pm} \notin E$ and G is claw-free, we have $y^-y^+ \in E$. Also, as yw^-, $uw^- \notin E$, vw^-, $yw^- \notin E$, and G is claw-free, we have $uy, vy \in E$. Thus, if $\sigma_3 = wy$, then there is a shorter chain $\sigma_0, \triangle_1, \sigma_1, \{v, w, y\}, wy = \sigma_3 P$ than P, a contradiction. If $\sigma_3 = xy$, then there is a shorter chain $\sigma_0, \{u, v, y\}, vy, \{v, x, y\}, xy = \sigma_3 P$ than P ,a contradiction.

Assume $y \notin V(C)$. As uw^-, $yw^- \notin E$, vx^-, $yx^- \notin E$, and G is claw-free, we have $uy, vy \in E$. Thus, if $\sigma_3 = wy$, then there is a shorter chain $\sigma_0, \{u, v, y\}, uy, \{u, w, y\}, wy = \sigma_3 P$ than P, a contradiction. If $\sigma_3 = xy$, then there is a shorter chain $\sigma_0, \{u, v, y\}, vy, \{v, x, y\}, xy = \sigma_3 P$ than P, a contradiction.

Case 2 $w \notin V(C)$

Let x be the first vertex of $V(C)$ appearing in the chain P, that is, $V(C) \cap \sigma_i = \emptyset$ for $i = 0, \cdots, k-1$ and $V(C) \cap \triangle_k \neq \emptyset$. We set $\sigma_{k-1} = st$ and $\triangle_k = \{s, t, x\}$. If $x = u$, then we can find a shorter chain than P. Thus we may assume that $x \neq u$. If $x = u^-$ or $x = u^+$, then we can find a longer cycle than C. Thus we may also assume that $x \neq u^{\pm}$. As G is claw-free, it follows that $x^-x^+ \in E$. In the following, we may assume that $\sigma_k = sx$, because the same argument holds for $\sigma_k = tx$. Let $\triangle_{k+1} = \{s, x, y\}$.

Case 2.1 $y \notin V(C)$

As tx^-, $yx^- \notin E$ and G is claw-free, we have $ty \in E$. If $\sigma_{k+1} = sy$, then there is a shorter chain $P\sigma_{k-1}, \{s, t, y\}, \sigma_{k+1}P$ than P, a contradiction. Thus, we may assume that $\sigma_{k+1} = xy$. Let $\triangle_{k+2} = \{x, y, z\}$.

Case 2.1.a $z \notin V(C)$

As tx^-, $zx^- \notin E$ and G is claw-free, we have $tz \in E$.

If $\sigma_{k+2} = xz$, then there is a shorter chain $P\sigma_{k-1}, \triangle_k, tx, \{t, x, z\}, xz = \sigma_{k+2}P$ than P, a contradiction.

If $\sigma_{k+2} = yz$, then there is a shorter chain $P\sigma_{k-1}, \{s, t, y\}, ty, \{t, y, z\}, yz = \sigma_{k+2}P$ than P, a contradiction.

Case 2.1.b $z \in V(C)$

As $yz^{\pm} \notin E$ and G is claw-free, we have $z^-z^+ \in E$. As tx^-, $zx^- \notin E$ and G is claw-free, we have $tz \in E$.

If $\sigma_{k+2} = xz$, then there is a shorter chain $P\sigma_{k-1}, \triangle_k, tx, \{t, x, z\}, xz = \sigma_{k+2}P$ than P, a contradiction.

If $\sigma_{k+2} = yz$, then there is a shorter chain $P\sigma_{k-1}, \{s, t, y\}, ty, \{t, y, z\}, yz = \sigma_{k+2}P$ than P, a contradiction.

Case 2.2 $y \in V(C)$

If $y = u$, then we can find a shorter chain than P. Thus we may assume that $y \neq u$. If $y = u^-$ or $y = u^+$, then we can find a longer cycle than C. Thus we may also assume that $y \neq u^{\pm}$. As $sy^{\pm} \notin E$ and G is claw-free, it follows that $y^-y^+ \in E$. As tx^-, $yx^- \notin E$ and G is claw-free, we have $ty \in E$. If $\sigma_{k+1} = sy$, then there is a shorter chain $P\sigma_{k-1}, \{s, t, y\}, \sigma_{k+1}P$ than P, a contradiction. Thus, we may assume that $\sigma_{k+1} = xy$. Let $\triangle_{k+2} = \{x, y, z\}$.

Case 2.2.a $z \notin V(C)$

As sy^-, $zy^- \notin E$ and G is claw-free, we have $sz \in E$. As tx^-, $zx^- \notin E$ and G is claw-free, we have $tz \in E$.

If $\sigma_{k+2} = xz$, then there is a shorter chain $P\sigma_{k-1}, \triangle_k, tx, \{t, x, z\}, xz = \sigma_{k+2}P$ than P, a contradiction.

If $\sigma_{k+2} = yz$, then there is a shorter chain $P\sigma_{k-1}, \{s, t, y\}$, sy, $\{s, y, z\}, yz = \sigma_{k+2}P$ than P, a contradiction.

Case 2.2.b $z \in V(C)$

As zy^-, $sy^- \notin E$ and G is claw-free, we have $sz \in E$. If $z = u$ or $z = u^+$, then this argument also holds. If $z = u-$, then take y^+ for y^- in this argument. As tx^+, $zx^+ \notin E$ and G is claw-free, we have $tz \in E$. If $z = u$ or $z = u^-$, then this argument also holds. If $z = u^+$, then take x^- for x^+ in this argument.

If $\sigma_{k+2} = xz$, then there is a shorter chain $P\sigma_{k-1}, \triangle_k, tx, \{t, x, z\}, xz = \sigma_{k+2}P$ than P, a contradiction.

If $\sigma_{k+2} = yz$, then there is a shorter chain $P\sigma_{k-1}, \{s, t, y\}, sy, \{s, y, z\}, yz = \sigma_{k+2}P$ than P, a contradiction.

This completes the proof of Theorem 2.

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