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On Explicit Formulas for the Dirichlet Divisor Problem and the Gauss Circle Problem

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Abstract

New explicit formulas for $\Delta(X)$ in the Dirichlet divisor problem and $P(X)$ in the Gauss circle problem are established. Although the formulas are variants of the celebrated Voronoï type formula, they are no more exponential sums but contain sine integral functions and furthermore, the expressions for $\Delta(X)$ and $P(X)$ have precisely the same appearance: denoting the arithmetical function $d(n)$ or $r(n)$ by $a(n)$, one has an explicit formula for $\Delta(X)$ or $P(X)$, up to the error term $O(X^{1/4+\delta/2+\varepsilon})$ with a constant δ satisfying $0<\delta<1$, expressed by a symmetric sum with the center $n=X$ whose summand is given by $a(n)\operatorname{sgn}(n-X)\pi^{-1}\operatorname{si}\left(\left|\sqrt{n}-\sqrt{X}\right|X^{\delta}\right)$.

1. Introduction and statement of the results

Let $d(n)$ denote the number of positive divisors of the integer n. Set

$$D(X)=\sum_{1\leq n\leq X}{}' d(n)=X(\log X+2\gamma-1)+1/4+\Delta(X), \tag{1.1}$$

where γ is the Euler constant and the symbol $\Sigma'_{1\leq n\leq X}$ denotes that the last term in the sum is halved if X is an integer. The Dirichlet divisor problem is to find the smallest value θ_0 of $\theta>0$ such that $\Delta(X)\ll X^{\theta+\varepsilon}$ holds for arbitrarily small positive constant ε.

Also let $r(n)$ denote the number of representations of the integer n as a sum of two squares. The Gauss circle problem is to find the smallest value θ_1 of $\theta>0$ such that $P(X)\ll X^{\theta+\varepsilon}$ holds, where $P(X)$ is defined by the equation

$$R(X)=\sum_{1\leq n\leq X}{}' r(n)=\pi X-1+P(X). \tag{1.2}$$

It is known that the omega theorems $\Delta(X)=\Omega(X^{1/4})$ and $P(X)=\Omega(X^{1/4})$ hold from the 'truncate formulas'

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(1.3) $$\Delta(X)=2^{-1/2}\pi^{-1}X^{1/4}\sum_{1\le n\le N}\frac{d(n)}{n^{3/4}}\cos\left(4\pi\sqrt{nX}-\pi/4\right)+O\left(X^{\varepsilon}+X^{1/2+\varepsilon}N^{-1/2}\right)$$

and

(1.4) $$P(X)=-\pi^{-1}X^{1/4}\sum_{1\le n\le N}\frac{r(n)}{n^{3/4}}\cos\left(2\pi\sqrt{nX}+\pi/4\right)+O\left(X^{\varepsilon}+X^{1/2+\varepsilon}N^{-1/2}\right).$$

These formulas are the trumcated form of the celebrated formulas, due to Voronoï and Hardy respectively, which represent $\Delta(X)$ and $P(X)$ by the infinite series with terms containing the Bessel functions (see (2.1) and (2.2) below). The expressions (1.3) and (1.4) for $\Delta(X)$ and $P(X)$ stimulated a lot of work on exponential sums in one or several variables. The relation $r(n)=4\sum_{d|n, d\equiv 1 \pmod 2}(-1)^{(d-1)/2}$ pointed out by Richert ([8]) shows that the circle problem is closely related to the divisor problem. As is seen in the excellent work due to Iwaniec-Mozzochi ([4]), the methods of estimating $\Delta(X)$ and $P(X)$ were up to now very similar, hence the obtained exponents for both were equal, although they were not always claimed explicitly in various publications. The famous conjecture $\theta_0=1/4$ or $\theta_1=1/4$ remains open as one of the most significant problems in number theory.

The aim of the present paper is to give, as variants of the Voronoï formula, new explicit formulas for $\Delta(X)$ and $P(X)$. Although they are variants of the Voronoï type formulas, they are no more exponential sums and the sine integral function $\mathrm{si}(x)=\int_{\infty}^{x}\frac{\sin t}{t}dt$ occurs:

Theorem. *For a constant δ satisfying $0<\delta<1$, one has*

(1.5) $$\Delta(X)=\frac{X^{3/4}}{\pi}\sum_{n=1}^{\infty}\mathrm{sgn}(n-X)\frac{d(n)}{n^{3/4}}\mathrm{si}\left(\left|\sqrt{n}-\sqrt{X}\right|X^{\delta}\right)+O\left(X^{1/4+\delta/2+\varepsilon}\right)$$

and

(1.6) $$P(X)=\frac{X^{3/4}}{\pi}\sum_{n=1}^{\infty}\mathrm{sgn}(n-X)\frac{r(n)}{n^{3/4}}\mathrm{si}\left(\left|\sqrt{n}-\sqrt{X}\right|X^{\delta}\right)+O\left(X^{1/4+\delta/2+\varepsilon}\right).$$

Note that these expressions for $\Delta(X)$ and $P(X)$ have precisely the same appearance: this may give another evidence for the resemblance of the two classical problems, or for the lattice point problem for the hybrid arithmetical function $b(n)=d(n)\pm r(n)$, the relevant error term can be expressed by

$$\frac{X^{3/4}}{\pi}\sum_{n=1}^{\infty}\mathrm{sgn}(n-X)\frac{b(n)}{n^{3/4}}\mathrm{si}\left(\left|\sqrt{n}-\sqrt{X}\right|X^{\delta}\right)$$

up to $O(X^{1/4+\delta/2+\varepsilon})$ error term.

Note also that the classical bounds $\Delta(X)\ll X^{1/3+\varepsilon}$ and $P(X)\ll X^{1/3+\varepsilon}$ follow also from the formulas (1.5) and (1.6).

Notation.

Throughout the paper, T stands for a large parameter and the abbreviation $L=\log T$ is frequently used. It will be convenient in the proofs to use the letters, c, c_k to denote certain positive numerical constants and, ε to denote a positive constant which may be arbitrarily small, but are not necessarily the same ones at each occurrence. For complex numbers z_1 and z_2, the symbol $[z_1, z_2]$ stands for the oriented segment from the point z_1 to z_2. For a real number $a\neq 0$, $\operatorname{sgn}(a)=a/|a|$.

2. The Voronoï formula and the Hardy formula

The Voronoï formula for $\Delta(X)$ is as follows (see, e.g., [3, Chapter 3]); one has

$$\Delta(X)=-\sqrt{X}\sum_{n=1}^{\infty}\frac{d(n)}{\sqrt{n}}\left\{Y_1\left(4\pi\sqrt{nX}\right)+\frac{2}{\pi}K_1\left(4\pi\sqrt{nX}\right)\right\} \tag{2.1}$$

where Y_1 is the ordinary Bessel function of the second kind and K_1 is the modified Bessel function of the second kind in usual notation and the series is boundedly convergent in any closed finite subinterval of the interval $(0, \infty)$, and is uniformly convergent in any such interval free from integers. The Hardy formula in the circle problem is as follows (see [3, Chapter 13]);

$$P(X)=\sqrt{X}\sum_{n=1}^{\infty}\frac{r(n)}{\sqrt{n}}J_1\left(2\pi\sqrt{nX}\right) \tag{2.2}$$

where J_1 is the ordinary Bessel function of the first kind. It holds the convergence assertion similar to that for (2.1). By using the asymptotic approximations involving elementary functions for Y_1- and K_1-Bessel functions (see, e.g., [3, (3.12), (3.13)]), the expression (2.1) is reduced to the exponential sums: for positive integer K, one has

$$\Delta(X)=\sum_{k=1}^{K}a_kX^{3/4-k/2}\sum_{n=1}^{\infty}\frac{d(n)}{n^{1/4+k/2}}\sin\left(4\pi\sqrt{nX}-(-1)^k\pi/4\right)+O\left(X^{1/4-K/2}\right) \tag{2.3}$$

with computable absolute constants a_k, whose first two terms being given by

$$a_1=1/(\sqrt{2}\pi) \quad\text{and}\quad a_2=-3/(32\sqrt{2}\pi^2). \tag{2.4}$$

As for $P(X)$, one has in a similar way,

$$P(X)=\sum_{k=1}^{K}b_kX^{3/4-k/2}\sum_{n=1}^{\infty}\frac{r(n)}{n^{1/4+k/2}}\sin\left(2\pi\sqrt{nX}+(-1)^k\pi/4\right)+O\left(X^{1/4-K/2}\right) \tag{2.5}$$

with computable absolute constants b_k, whose first two terms being given by

$$b_1=1/\pi \quad\text{and}\quad b_2=1/(16\pi^2). \tag{2.6}$$

3. Sine integral function and cosine integral function

To represent the exponential sums of the form (2.3), (2.5) by the sum with non-exponential terms, we use certain type of special functions which belong to incomplete gamma functions; namely, the sine integral function $\mathrm{si}(x)=\int_{\infty}^{x}\frac{\sin t}{t}dt$ and the cosine integral function $\mathrm{ci}(x)=\int_{\infty}^{x}\frac{\cos t}{t}dt$ (see, e.g., [2, Chapter IX]). In this section we collect some facts on the functions $\mathrm{si}(x)$ and $\mathrm{ci}(x)$, and give a simple lemma. From the definition one has

$$\mathrm{si}(x)=-\frac{\pi}{2}+\int_{0}^{x}\frac{\sin t}{t}dt \tag{3.1}$$

and

$$\mathrm{ci}(x)=r+\log x-\int_{0}^{x}\frac{1-\cos t}{t}dt. \tag{3.2}$$

From these, one has

$$\mathrm{si}(X)\ll X^{-1} \text{ and } \mathrm{ci}(X)\ll X^{-1}. \tag{3.3}$$

Next, we give an assertion on exponential integrals evaluated by sine and cosine integral functions.

Lemma. *For $A>0$, one has*

$$\int_{0}^{\infty}\frac{\exp(-Au\pm iA)}{1\pm iu}du=-\mathrm{si}(A)\pm i\mathrm{ci}(A). \tag{3.4}$$

Proof of Lemma. Firstly, note that from residue calculus by changing the variable x to ix, for $\alpha, \beta>0$,

$$\int_{0}^{\infty}\frac{\exp(-\alpha x)}{\beta^2+x^2}dx=\frac{1}{\beta}\left\{\mathrm{ci}(\alpha\beta)\sin(\alpha\beta)-\mathrm{si}(\alpha\beta)\cos(\alpha\beta)\right\} \tag{3.5}$$

holds. Secondly, by differentiating both sides of (3.5) as functions of α, one has

$$\int_{0}^{\infty}\frac{x\exp(-\alpha x)}{\beta^2+x^2}dx=-\mathrm{ci}(\alpha\beta)\cos(\alpha\beta)-\mathrm{si}(\alpha\beta)\sin(\alpha\beta). \tag{3.6}$$

Using these formulas by writing $\alpha=A$ and $\beta=1$, the left hand side of (3.4) becomes

$$\{\mathrm{ci}(A)\sin(A)\cos(A)\}\{\cos(A)\pm i\sin(A)\}$$
$$+\{-\mathrm{ci}(A)\cos(A)-\mathrm{si}(A)\sin(A)\}\{\sin(A)\mp i\cos(A)\},$$

and this is equal to the right hand side of (3.4).

4. Formulation of the proof

We shall dwell mainly on the proof of the formula (1.5) for $\Delta(X)$. Since the proof of the formula (1.6) for $P(X)$ can be carried out in much the same way as that of the formula (1.5), it will be mentioned briefly in the last section.

Starting point of the proof is the truncated formula (1.3) being deleted the first $(4\pi)^{-2}T^{2\delta}$ terms;

$$\Delta(T)=a_1T^{1/4}\sum_{\tau_1\leq n\leq\tau_2}\frac{d(n)}{n^{3/4}}\cos\left(4\pi\sqrt{nT}-\pi/4\right)+O\left(T^{1/4+\delta/2+\varepsilon}\right), \tag{4.1}$$

where $a_1=2^{-1/2}\pi^{-1}$, $\tau_1=(4\pi)^{-2}T^{2\delta}$ and $\tau_2=(4\pi)^{-2}T^2$. Put

$$V(T)=a_1T^{1/4}\sum_{\tau_1\leq n\leq\tau_2}\frac{d(n)}{n^{3/4}}\exp\left(4\pi i\sqrt{nT}-\pi i/4\right). \tag{4.2}$$

By using (1.1), $V(T)$ is transformed, up to $O(T^{1/4})$ error term, into

$$\begin{aligned} &a_1T^{1/4}\int_{\tau_1}^{\tau_2}X^{-3/4}\exp\left(4\pi i\sqrt{XT}-\pi i/4\right)\left(\log X+2r\right)dX \\ &+a_1T^{1/4}\left[X^{-3/4}\exp\left(4\pi i\sqrt{XT}-\pi i/4\right)\Delta(X)\right]_{\tau_1}^{\tau_2} \\ &-a_1T^{1/4}\int_{\tau_1}^{\tau_2}\left\{-(3/4)X^{-7/4}+2\pi i\sqrt{T}X^{-5/4}\right\}\exp\left(4\pi i\sqrt{XT}-\pi i/4\right)\Delta(X)dX. \end{aligned} \tag{4.3}$$

By changing the variable X to X^2, the first term in (4.3) is estimated by $O(T^{\varepsilon})$ from the first derivative test. The second term is estimated by $O(T^{1/4})$. The last term with the integrand $-(3/4)X^{-7/4}\exp(4\pi i\sqrt{XT}-\pi i/4)\Delta(X)$ is estimated by $O(T^{1/4+\varepsilon})$. Hence we have

$$V(T)=-\sqrt{2}\,iT^{3/4}\int_{\tau_1}^{\tau_2}X^{-5/4}\exp\left(4\pi i\sqrt{XT}-\pi i/4\right)\Delta(X)dX+O(T^{1/4+\varepsilon}). \tag{4.4}$$

In substituting the exponential expression (2.3) for $\Delta(X)$ in (4.4), if we take $K=[1/(2\delta)]+1$ series in (2.3), where the symbol $[x]$ stands for the greatest integer not greater than x, the error term $O(X^{1/4-K/2})$ contributes to $V(T)$ an amount $O(T^{1/4}\log T)$. For $2\leq k\leq K$, the series in the formula (2.3) takes the form

$$c_kT^{3/4}\sum_{n=1}^{\infty}\frac{d(n)}{n^{1/4+k/2}}\int_{\tau_1}^{\tau_2}\frac{1}{X^{1/2+k/2}}\exp\left\{4\pi i(\sqrt{T}\pm\sqrt{n})\sqrt{X}-\pi i/4\mp(-1)^k\pi i/4\right\}dX.$$

By changing the variable X to X^2, integrals in the series are estimated by $O(T^{-\delta}|\sqrt{T}\pm\sqrt{n}|^{-1})$ with $n\neq T$. These contribute to $V(T)$ an amount $O(T^{1/4})$.

Thus, we are left with the series for $k=1$:

$$(4.5)\quad V(T)=-\frac{iT^{3/4}}{\pi}\sum_{n=1}^{\infty}\frac{d(n)}{n^{3/4}}\int_{\tau_1}^{\tau_2}\frac{1}{X}\exp\Big(4\pi i\sqrt{XT}-\pi i/4\Big)\cos\Big(4\pi\sqrt{nX}-\pi/4\Big)dX+O(T^{1/4+\varepsilon}).$$

Changing the variable $4\pi\sqrt{X}$ by X, up to $O(T^{1/4})$ error term, one has

$$(4.6)\qquad V(T)=V_{+}(T)+V_{-}(T)+O(T^{1/4+\varepsilon})$$

where

$$(4.7)\qquad V_{+}(T)=-\pi^{-1}T^{3/4}\sum_{n=1}^{\infty}\frac{d(n)}{n^{3/4}}\int_{T^{\delta}}^{T}X^{-1}\exp\Big(i(\sqrt{T}+\sqrt{n})X\Big)dX$$

and

$$(4.8)\qquad V_{-}(T)=-i\pi^{-1}T^{3/4}\sum_{n=1}^{\infty}\frac{d(n)}{n^{3/4}}\int_{T^{\delta}}^{T}X^{-1}\exp\Big(i(\sqrt{T}-\sqrt{n})X\Big)dX.$$

5. Estimation of the series $V_{+}(T)$

Note that the sum of the terms with $n>T^2$ is estimated by $O(T^{1/4+\varepsilon})$. For the terms with $n\leq T^2$, putting $u_1=(\sqrt{T}+\sqrt{n})^{-1}L^2$ where $L=\log T$, we change the contour $[T^{\delta}, T]$ of the integrals in the series $V_{+}(T)$ to $C_1+C_2+C_3$ where $C_1=[T^{\delta}, T^{\delta}(1+iu_1)]$, $C_2=[T^{\delta}(1+iu_1), T(1+iu_1)]$ and $C_3=[T(1+iu_1), T]$. On the contour C_1 and C_3, the variable X is changed into u with $0\leq u\leq u_1$ by $X=x(1+iu)$ where $x=T^{\delta}$ or T. Integrals on the contours C_2 and C_3 are estimated by $O(\exp(-c\,L^2))$ and $O(n^{-1/2}T^{-1})$ respectively. These contribute to $V_{+}(T)$ an amount $O(1)$. Since the integrals on the contour C_1 is equal to

$$i(\sqrt{T}+\sqrt{n})^{-1}T^{-\delta}\exp\Big(i(\sqrt{T}+\sqrt{n})T^{\delta}\Big)+O\Big((\sqrt{T}+\sqrt{n})^{-2}T^{-2\delta}\Big),$$

denoting the sum of the terms with $n\leq T^2$ in the series $V_{+}(T)$ by Σ, one has,

$$(5.1)\qquad \Sigma=\pi^{-1}T^{3/4-\delta}\sum_{1\leq n\leq T^2}\frac{d(n)}{n^{3/4}}(\sqrt{T}+\sqrt{n})^{-1}\exp\Big(i(\sqrt{T}+\sqrt{n})T^{\delta}\Big)+O(T^{1/4}).$$

To estimate the sum Σ, by using (1.1) we transform it further;

$$\begin{aligned}(5.2)\qquad \Sigma=&\,\pi^{-1}T^{3/4-\delta}\int_1^{T^2}Y^{-3/4}(\sqrt{T}+\sqrt{Y})^{-1}\exp\Big(i(\sqrt{T}+\sqrt{Y})T^{\delta}\Big)(\log Y+2r)dY\\ &+\pi^{-1}T^{3/4-\delta}\Big[Y^{-3/4}(\sqrt{T}+\sqrt{Y})^{-1}\exp\Big(i(\sqrt{T}+\sqrt{Y})T^{\delta}\Big)\Delta(Y)\Big]_1^{T^2}\\ &-\pi^{-1}T^{3/4-\delta}\int_1^{T^2}\Big\{-(3/4)Y^{-7/4}-2^{-1}Y^{-3/4}(\sqrt{T}+\sqrt{Y})^{-1}+i2^{-1}T^{\delta}Y^{-5/4}\Big\}\\ &\qquad\times(\sqrt{T}+\sqrt{Y})^{-1}\exp\Big(i(\sqrt{T}+\sqrt{Y})T^{\delta}\Big)\Delta(Y)dY+O(T^{1/4}).\end{aligned}$$

The first and the second terms in (5.2) are estimated by $O(T^{1/4})$. In the third term, the integrals with the integrand

$$\left\{-(3/4)Y^{-7/4}-2^{-1}Y^{-3/4}(\sqrt{T}+\sqrt{Y})^{-1}\right\}(\sqrt{T}+\sqrt{Y})^{-1}\exp\left(i(\sqrt{T}+\sqrt{Y})T^{\delta}\right)\Delta(Y)$$

are estimated by $O(T^{-1/2})$. To estimate the integral

$$-i2^{-1}\pi^{-1}T^{3/4}\int_1^{T^2} Y^{-5/4}(\sqrt{T}+\sqrt{Y})^{-1}\exp\left(i(\sqrt{T}+\sqrt{Y})T^{\delta}\right)\Delta(Y)dY, \tag{5.3}$$

we use the formula (2.3) for $\Delta(Y)$ in the case $K=1$;

$$\Delta(Y)=a_1 Y^{1/4}\sum_{n=1}^{\infty}\frac{d(n)}{n^{3/4}}\cos\left(4\pi\sqrt{nY}-\pi/4\right)+O(Y^{-1/4}). \tag{5.4}$$

The error term $O(Y^{-1/4})$ contribute to (5.3) an admissible error $O(T^{1/4})$. The integral in (5.3), substituted by the first term in (5.4), takes the form

$$cT^{3/4}\sum_{n=1}^{\infty}\frac{d(n)}{n^{3/4}}\int_1^{T} Y^{-1}(\sqrt{T}+Y)^{-1}\exp\left\{iT^{1/2+\delta}+i(T^{\delta}\pm4\pi\sqrt{n})Y\mp\pi i/4\right\}dY. \tag{5.5}$$

Since the series of integral terms with the integrand

$$Y^{-1}(\sqrt{T}+Y)^{-1}\exp\{iT^{1/2+\delta}+i(T^{\delta}+4\pi\sqrt{n})Y-\pi i/4\}$$

is estimated by $O(T^{1/4})$ and the series with the integrand

$$Y^{-1}(\sqrt{T}+Y)^{-1}\exp\{iT^{1/2+\delta}+i(T^{\delta}-4\pi\sqrt{n})Y+\pi i/4\}$$

is estimated by $O(T^{1/4+\delta/2+\varepsilon})$, $V_+(T)$ in (4.6) is estimated by $O(T^{1/4+\delta/2+\varepsilon})$.

6. Completion of the proof: evaluation of the series $V_-(T)$

It remains for us to evaluate the series $V_-(T)$. The sum of the terms with $n>T^2$ is estimated by $O(T^{1/4+\varepsilon})$. We may suppose that $n\neq T$ with an admissible error term. Let us denote the sum of the terms with $n\leq T-1$ by Σ_1 and those with $T+1\leq n\leq T^2$ by Σ_2.

For the integral terms in Σ_1, we change the contour $[T^{\delta}, T]$ of the integrals to $C_1+C_2+C_3$ where, putting $u_1=(\sqrt{T}-\sqrt{n})^{-1}T^{-\delta}L^2$ and $u_2=(\sqrt{T}-\sqrt{n})^{-1}T^{-1}L^2$, $C_1=[T^{\delta}, T^{\delta}(1+iu_1)]$, $C_2=[T^{\delta}(1+iu_1), T(1+iu_2)]$ and $C_3=[T(1+iu_2), T]$. On the contour C_1 and C_3, the variable X is changed into u with $0\leq u\leq u_1$ or u_2 by $X=x(1+iu)$ where $x=T^{\delta}$ or T. Thus, by using the formula (3.4) for $A=(\sqrt{T}-\sqrt{n})T^{\delta}$, the integrals on the contour C_1 is equal to

$$(6.1)\qquad i\int_0^\infty \exp\left\{i(\sqrt{T}-\sqrt{n})T^\delta-(\sqrt{T}-\sqrt{n})T^\delta u\right\}(1+iu)^{-1}du+O\Big(\exp(-cL^2)\Big)$$
$$=i\Big\{-\mathrm{si}\Big((\sqrt{T}-\sqrt{n})T^\delta\Big)+i\mathrm{ci}\Big((\sqrt{T}-\sqrt{n})T^\delta\Big)\Big\}+O\Big(\exp(-cL^2)\Big).$$

Also the integrals on the contour C_3 are equal to

$$-i\Big\{-\mathrm{si}\Big((\sqrt{T}-\sqrt{n})T\Big)+i\mathrm{ci}\Big((\sqrt{T}-\sqrt{n})T\Big)\Big\}+O\Big(\exp(-cL^2)\Big),$$

and by using (3.3), these contribute to $V(T)$ an amount $O(1)$. Contribution from the contour C_2 is very small. Thus, one has

$$(6.2)\qquad \Sigma_1=-\pi^{-1}T^{3/4}\sum_{1\le n\le T-1}\frac{d(n)}{n^{3/4}}\Big\{\mathrm{si}\Big((\sqrt{T}-\sqrt{n})T^\delta\Big)-i\mathrm{ci}\Big((\sqrt{T}-\sqrt{n})T^\delta\Big)\Big\}+O(1).$$

As for the integral terms in Σ_2, we change the contour $[T^\delta, T]$ into $C_1'+C_2'+C_3'$ where, putting $u_1=(\sqrt{n}-\sqrt{T})^{-1}T^{-\delta}L^2$ and $u_2=(\sqrt{n}-\sqrt{T})^{-1}T^{-1}L^2$, $C_1'=[T^\delta, T^\delta(1-iu_1)]$, $C_2'=[T^\delta(1-iu_1), T(1-iu_2)]$ and $C_3'=[T(1-iu_2), T]$. Then, by an argument similar to that in the case for Σ_1, one has

$$(6.3)\qquad \Sigma_2=\pi^{-1}T^{3/4}\sum_{T+1\le n\le T^2}\frac{d(n)}{n^{3/4}}\Big\{\mathrm{si}\Big((\sqrt{n}-\sqrt{T})T^\delta\Big)+i\mathrm{ci}\Big((\sqrt{n}-\sqrt{T})T^\delta\Big)\Big\}+O(1).$$

Taking the real part of the series (6.2) and (6.3), combined with (4.1), (4.2), (4.5) and (4.8), and using (3.3) for the terms in (1.5) with $n>T^2$, we complete the proof of the formula (1.5).

The classical estimation $\Delta(X)\ll X^{1/3+\varepsilon}$ follows from (3.3), by taking $\delta=1/6$ in the formula (1.5).

As for the circle problem, the sum

$$-\pi^{-1}T^{1/4}\sum_{\tau_1\le n\le\tau_2} r(n)n^{-3/4}\exp\Big(2\pi i\sqrt{nT}+\pi/4\Big)$$

with $\tau_1=(2\pi)^{-2}T^{2\delta}$ and $\tau_2=(2\pi)^{-2}T^2$ in the truncated formula (1.4) is transformed, by using (1.2) with (2.5), into the series

$$\pi^{-1}T^{3/4}\sum_{n=1}^\infty\frac{r(n)}{n^{3/4}}\int_{T^\delta}^T X^{-1}\exp\Big(i(\sqrt{T}+\sqrt{n})X\Big)dX$$
$$-i\pi^{-1}T^{3/4}\sum_{n=1}^\infty\frac{r(n)}{n^{3/4}}\int_{T^\delta}^T X^{-1}\exp\Big(i(\sqrt{T}-\sqrt{n})X\Big)dX,$$

which corresponds to the steps (4.7) and (4.8). From these, one has the formula (1.6) in much the same way as the proof of the formula (1.5).

Remarks.

(1) Needless to say, the infinite series expression of the formulas (1.5) and (1.6) is not essential. The principal part of our explicit formulas consists in the symmetric sine integral sum with the center X: by investigating into the sine integral sum in the form of

$$\mathcal{S}(T^{\delta}, \alpha T^{a}) = \sum_{T-\alpha T^{a} \leq n \leq T+\alpha T^{a}} a(n)\operatorname{sgn}(n-T)\pi^{-1}\operatorname{si}\left(4\pi\left|\sqrt{n}-\sqrt{T}\right|T^{\delta}\right),$$

where $a(n)=d(n)$ or $r(n)$, one may shorten the length of the series in Theorem to $c\sqrt{T}$ for some c (cf. [7]).

(2) In the duality between $d(n)$ and $|\zeta(\frac{1}{2}+iT)|^2$, the Voronoï phase function $4\pi\sqrt{nX}-\pi/4$ in (1.3) corresponds to the Atkinson function

$$2T\operatorname{arsinh}\sqrt{\pi n/(2T)}+2\pi n\sqrt{1/4+T/(2\pi n)}+\pi/4$$

in the Atkinson formula or the Jutila formula for $|\zeta(\frac{1}{2}+iT)|^2$ ([1], [5]). The expression by the function $\operatorname{sgn}(n-T)\operatorname{si}\left(\left|\sqrt{n}-\sqrt{T}\right|T^{\delta}\right)$ with center T in the formula (1.5) seems to correspond to that with the phase function

$$2T\operatorname{arcosh}\sqrt{\pi n/(2T)}-2\pi n\sqrt{1/4-T/(2\pi n)}+\pi/4$$

occurred in the alternative explicit formula for $|\zeta(\frac{1}{2}+iT)|^2$ (see [6]).

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