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On Fundamental Formulas of Foliations.

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Abstract

In recent years, there have been several studies of foliations from differential geometric aspects. Especially, many differential geometric properties of metric foliations have been studied. In those studies, O'Neill's fundamental equations played a central role. These equations are derived in the study of differential geometric properties of Riemannian submersions, which is defined by O'Neill and is a special class of metric foliations, which we do not treat in this paper. For general foliations, many results are also obtained. However, the approach is done from various view points depending on the mathematicians and, at present, there seems to be no systematic approach.

In this paper, we present some fundamental formulas for the differential geometric study of arbitrary foliations. These formulas turns out to be useful especially when curvature conditions are given. We give new proofs of many known results from this view points, and also give some new results.

1. Introduction

In recent years, there have been several studies of foliations from differential geometric aspects (cf. Reinhart [Rh]). Especially, many differential geometric properties of metric foliations have been studied (cf. Tondeur [Td]). In those studies, O'Neill's fundamental equations [ON] played a central role. These equations are derived in the study of differential geometric properties of Riemannian submersions, which is defined by O'Neill and is a special class of metric foliations, which we do not treat in this paper. For general foliations, many results are also abtained. However, the approach is done from various view points depending on the mathematicians and, at present, there seems to be no systematic approach.

In this paper, we present some fundamental formulas for the differential geometric

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study of arbitrary foliations. These formulas turns out to be useful especially when curvature conditions are given. We give new proofs of many known results from this view point, and also give some new results. In §2, some preliminaries and the first fundamental formula [FF. I] are given. [FF. I] plays the central role in this paper. In §3, codimension-one foliations of compact manifolds are studied by means of the second fundamental formula [FF. II]. Theorems 1 and 2 are extension of the author's results from closed manifolds to compact manifolds with boundary. In §4, foliations of codimension greater than one are studies by means of the third and forth fundamental formulas, [FF. III] and [FF. IV]. Theorems 5 and 9 are reproved from our view point. It is also pointed out that Ranjan's results [Rj] are closely related to our study. In §5, we consider foliations of complete manifolds, and some new results are given.

2. Preliminaries

Let (M,g) be a Riemannian manifold. Set $\dim(M)=n$. Denote by ∇ the Riemannian connection of (M, g). The curvature tensor R of (M, g) is defined by

$$R(X, Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[Y, X]}Z$$

for $X, Y, Z \in TM$. Let π be the 2-plane in T_xM spanned by linearly independent vectors $X, Y \in T_xM$. The sectional curvature K_π is defined by

$$K_\pi=\frac{\langle R(X, Y)Y, X\rangle}{|X|^2|Y|^2-\langle X, Y\rangle^2} .$$

Here $\langle X, Y\rangle$ means $g(X,Y)$. If X, Y are unit vectors and perpendicular, we denote K_π by $K(X, Y)$. A Riemannian manifold (M, g) satisfying $K_\pi\geq 0$ (resp. ≤ 0) for all 2-planes π in T_xM, $x \in M$, is called a Riemannian manifold of non-negative (resp. non-positive) sectional curvature. Let $\{V_1, V_2, \cdots, V_n\}$ be an orthonormal frame of TM. The Ricci curvature tensor $\mathrm{Ric}(X, Y)$ of (M, g) is defined by

$$\mathrm{Ric}(X, Y)=\sum_{i=1}^{n}\langle R(V_i, X)Y, V_i\rangle$$

for $X, Y \in TM$. A Riemannian manifold (M, g) satisfying $\mathrm{Ric}(X, Y)\geq 0$ for all $X, Y \in TM$ is called a Riemannian manifold with non-negative Ricci curvature. The scalar curvature S of (M, g) is defined by

$$S=\sum_{i=1}^{n}\mathrm{Ric}(V_i, V_i).$$

Let X be a vector field on M. The divergence $\mathrm{div}(X)$ of X is defined by

$$\mathrm{div}(X)=\sum_{i=1}^{n}\langle\nabla_{V_i}X, V_i\rangle.$$

Also define a linear map $D_X\colon T_xM \to T_xM$, $x \in M$, by $D_X(V)=\nabla_V X$, $V \in T_xM$. Note that $\mathrm{div}(V)=\mathrm{Tr}(D_X)$, where $\mathrm{Tr}(D_X)$ is the trace of the linear map D_X. The following is a well-known formula (cf. Kobayashi and Nomizu [KN]).

Theorem 0.1 (Green's Theorem) *If $\nabla_X X$ is tangent to ∂M in case of $\partial M \neq \emptyset$, we have*

$$\int_M \{\mathrm{Ric}(X, X) + \mathrm{Tr}(D_X^2) + X(\mathrm{div}(X))\} = 0,$$

where we omit the volume element dV of (M, g) for simplicity.

Let F be a codimension-q foliation of a Riemannian manifold (M, g). Set $p = \dim(F) = n-q$, and H the orthogonal complement of F in TM. We also use the same symbols F and H to denote the natural projections $F: TM \to F$ and $H: TM \to H$. Define two tensors A and T of type $(1, 2)$ by

$$T_V W = H\nabla_{FV} FW + F\nabla_{FV} HW$$

and

$$A_V W = H\nabla_{HV} FW + F\nabla_{HV} HW$$

for $V, W \in TM$.

The mean curvature $\boldsymbol{H}$ of F is defined by

$$\boldsymbol{H} = \sum_{i=1}^{p} T_{Ei} Ei = H(\sum_{i=1}^{p} \nabla_{Ei} E_i),$$

where $\{E_1, E_2 \cdots E_p\}$ is a local orthonormal frame of F. F is said to be minimal if $\boldsymbol{H} \equiv 0$, and said to be totally geodesic if $T \equiv 0$. It is well-known that if F is totally geodesic and H is integrable and totally geodesic, then (M, g) is locally isometric to the Riemannian product of $(L, g|L)$ and $(K, g|K)$ for $L \in F$ and $K \in H$ (see Kobayashi and Nomizu [KN]). In this case, we say (M, g) is locally a product of F and H. For the generalities on differential geometric aspects of foliations, see the books of B. Reinhardt [Rh] and P. Tondeur [Td].

Now we give a formula, which plays a central role in this paper. This is given by calculating $K(E, X)$ for unit vectors $E \in F$ and $X \in H$ as follows.

$$K(E, X) = \langle \nabla_E \nabla_X X, E\rangle - \langle \nabla_X \nabla_E X, E\rangle - \langle \nabla_{[E, X]} X, E\rangle = \langle \nabla_E \nabla_X X, E\rangle - X\langle \nabla_E X, E\rangle$$
$$+ \langle \nabla_E X, \nabla_X E\rangle - \Sigma_{i=1}^{p} \langle [E, X], E_i\rangle \langle \nabla_{E_i} X, E\rangle - \Sigma_{\alpha=1}^{q} \langle [E, X], X_\alpha\rangle \langle \nabla_{X_\alpha} X, E\rangle,$$

where $\{E_1, E_2, \cdots E_p\}$ is a local orthonormal frame of F and $\{X_1, X_2, \cdots, X_q\}$ is a local orthonormal frame of H.

Thus we have

$$[\mathrm{FF.\,I}] \quad K(E, X)$$
$$= \langle \nabla_E \nabla_X X, E\rangle - X\langle T_E E, X\rangle - |T_E X|^2 - \sum_{\alpha=1}^{q} \langle A_X X_\alpha, E\rangle \langle A_{X_\alpha} X, E\rangle$$
$$+ 2\sum_{i=1}^{p} \langle \nabla_E X, E_i\rangle \langle E_i, \nabla_X E\rangle - \sum_{\alpha=1}^{q} \langle \nabla_E X, X_\alpha\rangle \langle \nabla_{X_\alpha} X, E\rangle$$

$$-\sum_{\alpha=1}^{q}\langle\nabla_E X, X_\alpha\rangle\langle\nabla_X X_\alpha, E\rangle.$$

In the following sections, we sum up [FF. I], $K(E_i, X_\alpha)$, for suitable E_i's and X_α's in order to get fundamental formulas applicable to our situations.

3. Case of codimension one

In this section, we assume that (M, g, F) is a codimension-one foliation of a compact connected Riemannian manifold (M, g), and that the boundary components are leaves of F if $\partial M \neq \emptyset$. As the results presented in this paper are valid if we lift everything onto a finite covering space of M, we may assume M and F are oriented without loss of generality. Hence we can choose a unit vector field N on M perpendicular to F everywhere on M so that the orientation of M coincides with the one given by F and N. Note that, as the dimension of H is one, H is always integrable and the leaves of H are the orbits of N. For simplicity of our calculation, we set $\dim(M)=n+1$, thus $\dim(F)=n$.

Let $\{E_1, E_2, \cdots, E_n\}$ be a local orthonormal frame of F. Summing up [FF. I], $K(E_i, N)$, for $i=1, 2, \cdots, n$, we have the second fundamental formula [FF. II], which is similar to Green's theorem for taking $X=N$. Set $h=\langle \boldsymbol{H}, N\rangle$.

$$\text{[FF. II]} \qquad \mathrm{Ric}(N, N)+\mathrm{Tr}(T_N^2)+N(h)=\mathrm{div}(\nabla_N N)$$

To show this, we have only to note that $\Sigma_{i,j=1}^{n}\langle\nabla_{E_i}N, E_j\rangle\langle E_i, \nabla_N E_j\rangle=0$, because $\langle\nabla_{E_i}N, E_j\rangle$ is symmetric in i and j, but $\langle E_i, \nabla_N E_j\rangle$ is skew-symmetric in i and j.

Using [FF. II], we have the following results.

Theorem 1 ([O1]) *Let (M, g, F) be a codimension-one minimal foliation. If (M, g) is of non-negative Ricci curvature, then F is totally geodesic. Further, N is a parallel vector field. Hence, (M, g) is locally a product of F and H.*

(*Proof*) As $\nabla_N N$ is tangent to ∂M if $\partial M \neq \emptyset$, we have the following by integrating [FF. II].

$$\int_M \{\mathrm{Ric}(N, N)+\mathrm{Tr}(T_N^2)+N(h)\}=0$$

By assumption, $h\equiv 0$, $\mathrm{Ric}(N, N)\geq 0$ and $\mathrm{Tr}(T_N^2)\geq 0$. It follows that $\mathrm{Ric}(N, N)=0$ and $\mathrm{Tr}(T_N^2)=0$. In particular, F is totally geodesic.

To show the local productness of (M, g), we have only to show that $\nabla_N N=0$. First note that $\mathrm{div}(\nabla_N N)=0$ by[FF. II], and that $\nabla_{\nabla_N N}\nabla_N N$ is tangent to ∂M as F is totally geodesic. Applying Green's theorem for $X=\nabla_N N$, we have

$$\int_M \{\mathrm{Ric}(\nabla_N N, \nabla_N N)+\mathrm{Tr}(D^2_{\nabla_N N})\}=0.$$

As the dual one-form θ of $\nabla_N N$ is closed on each leaf of F, we have

$$\begin{aligned}\mathrm{Tr}(D^2_{\nabla_N N}) &= \sum_{i=1}^{n}\langle D^2_{\nabla_N N}(E_i), E_i\rangle + \langle D^2_{\nabla_N N}(N), N\rangle \\ &= \sum_{i=1}^{n}\langle \nabla_{\nabla_{E_i}\nabla_N N}\nabla_N N, E_i\rangle + \langle \nabla_{\nabla_N\nabla_N N}\nabla_N N, N\rangle \\ &= \sum_{i,j=1}^{n}\langle \nabla_{E_i}\nabla_N N, E_j\rangle^2 + \langle \nabla_N\nabla_N N, N\rangle^2 \geq 0\end{aligned}$$

By assumption, $\mathrm{Ric}(\nabla_N N, \nabla_N N)\geq 0$. Thus $\mathrm{Tr}(D^2_{\nabla_N N})=0$. In particular, $0=\langle \nabla_N\nabla_N N, N\rangle = -|\nabla_N \nabla N|^2$. This completes the proof.

Theorem 2 ([O1]) *Let (M, g, F) be a codimension-one totally geodesic foliation. If (M, g) is of non-positive sectional curvature, then N is a parallel vector field. Hence, (M, g) is locally a product of F and H.*

(*Proof.*) In this case, [FF. I] becomes

$$K(E, N)=\langle \nabla_E\ \nabla_N N, E\rangle - |\nabla_N N|^2.$$

By polarization, we have

$$\nabla_E \nabla_N N = \langle \nabla_N N, E\rangle \nabla_N N \quad \textit{for } E\in F$$

This formula enables us to show $\mathrm{Ric}(\nabla_N N, \nabla_N N)=0$. Thus, by the same argument as in the previous theorem, we have $\nabla_N N=0$, which completes the proof.

Theorem 3 ([BW]) *Let (M, g, F) be a codimension-one foliation of a closed Riemannian manifold with non-negative Ricci curvature. If the orbits of N are geodesics, then F is also totally geodesic. Hence, (M, g) is locally a product of F and H.*

(*Proof.*) It suffices to prove that F is totally geodesic. In this case, [FF. II] becomes

$$\mathrm{Ric}(N, N)+\mathrm{Tr}(T_N^2)+N(h)=0,$$

because H is totally geodesic, thus $\nabla_N N=0$. As h is a smooth function on a closed manifold M, h attains a maximum value at, say, $x\in M$. Thus $N(h)(x)=0$. As $\mathrm{Ric}(N, N)\geq 0$ and $\mathrm{Tr}(T_N^2)\geq 0$, the above formula implies $\mathrm{Tr}(T_N^2)(x)=0$. This means $h(x)=0$. Therefore, we have $h\leq 0$. The same argument at a minimum value gives $h\geq 0$. By these two inequalities, we have $h\equiv 0$, that is, F is a minimal foliation. By Theorem 1, we have the desired conclusion.

We say that a codimension-one foliation F is a constant mean curvature foliation if the mean curvature $\boldsymbol{H}$ of F is constant on each leaf of F. We can extend Theorem 1 for closed manifolds as follows.

Theorem 4 ([BKO]) *Let (M, g, F) be a codimension-one constant mean curvature foliation of a closed Riemannian manifold. If the Ricci curvature of (M, g) is non-negative, then F is totally geodesic. Hence, N is parallel and (M, g) is locally a product of F and H.*

(*Proof.*) To prove this, we need a qualitative property of codimension-one foliations (for the proof, see [BKO]).

Set $S=\{x\in M|h$ attains its maximum value at $x\}$. If h is not constant on M, then S contains a compact leaf of F.

If h is constant on M, then $N(h)\equiv 0$ on M. Thus the same argument as in the proof of Theorem 1 gives the desired conclusion.

Now assume h is not constant on M. Then $S(\neq\emptyset)$ contains a compact leaf, say, $L\in F$. On L, [FF. II] becomes

$$\mathrm{Ric}(N, N)+\mathrm{Tr}(T_N^2)+|\nabla_N N|^2=\mathrm{div}_L(\nabla_N N),$$

where $\mathrm{div}_L(\nabla_N N)$ is the divergence of the vector field $\nabla_N N$ on the Riemannian manifold $(L, g|L)$. By integrating this formula over L, we have

$$\int_L\{\mathrm{Ric}(N, N)+\mathrm{Tr}(T_N^2)+|\nabla_N N|^2\}=0.$$

As each term of the integrand is non-negative, we have, in particular, $\mathrm{Tr}(T_N^2)=0$ on L, that is, L is totally geodesic. Thus $h=0$ on L. This means $h\leq 0$ on M, because h attains its maximum value on L. The same argument on the set, where h attains its minimum value, gives $h\geq 0$. Therefore, we have $h\equiv 0$, that is, F is minimal. Now, by Theorem 1, this completes the proof.

Note that Theorems 3 and 4 do not hold if $\partial M\neq\emptyset$. In fact, concentric spheres in Euclidean spaces give counter examples.

4. Case of codimension ≥ 2

In the previous section, we show that codimension-one minimal foliations are totally geodesic if (M, g) is of non-negative Ricci curvature. This does not hold if the codimension of a foliation is greater than one. In fact, Takagi and Yorozu [TY] gave such examples with more restricted conditions.

Theorem 0.2 ([TY]) *There are examples of minimal, but non-totally geodesic foliations F on Lie groups with non-negative Ricci curvature. Further, the metric can be chosen to be bundle-like with respect to F.*

In this section, we mainly try to extend the results in §3 for foliations of codimension greater than one by using [FF. I]. We assume that (M, g, F) is a codimension-q foliation of a closed connected Riemannian manifold (M, g). As the results presented in this paper are valid if we lift everything onto a finite covering space of M, we may assume M and F are oriented without loss of generality.

Let $\{E_1, E_2, \cdots, E_p\}$ and $\{X_1, X_2, \cdots, X_q\}$ be local orthonormal frames of F and H. For a unit (locally defined) vector field $X \in H$, by summing up [FF. I], $K(E_i, X)$, for $i=1, 2, \cdots, p$, we have

$$[\text{FF. III}] \quad \sum_{i=1}^{p} K(E_i, X) = \sum_i \langle \nabla_{E_i} \nabla_X X, E_i \rangle + X\langle H, X\rangle - \sum_i |T_{E_i} X|^2 - \sum_\alpha \langle A_X X_\alpha, A_{X_\alpha} X\rangle - 2\sum_{i,\alpha} \langle \nabla_{E_i} X, X_\alpha \rangle \langle E_i, \nabla_X X_\alpha \rangle.$$

Here, the summation Σ_i is taken for $i=1, 2, \cdots, p$, and Σ_α is taken for $\alpha=1, 2, \cdots, q$. Hereafter the indices i, j, k indicate the values $1, 2, \cdots, p$ and the indices α, β, γ indicate the values $1, 2, \cdots, q$ when they appear in summations.

Theorem 5 ([Br]) *Let (M, g, F) be a codimension-two minimal foliation of a closed Riemannian manifold with non-negative Ricci curvature. If H is integrable and trivial, then F is totally geodesic.*

(*Proof.*) As H is trivial, we can choose global unit vector fields $X, Y \in H$ with $\langle X, Y\rangle = 0$. Note that, as $\dim(H)=2$, $\{X, Y\}$ is a global orthonormal frame of H. First we calculate $K(X, Y)$.

$K(X, Y) = K(Y, X) = \langle \nabla_Y \nabla_X X, Y\rangle - \langle \nabla_X \nabla_Y X, Y\rangle - \langle \nabla_{[Y, X]} X, Y\rangle = \langle \nabla_Y \nabla_X X, Y\rangle - X\langle Y, \nabla_Y X\rangle + \langle \nabla_Y X, \nabla_X Y\rangle - \langle [Y, X], X\rangle \langle \nabla_X X, Y\rangle - \langle [Y, X], Y\rangle \langle \nabla_Y X, Y\rangle = \langle \nabla_Y \nabla_X X, Y\rangle + X\langle X, \nabla_Y Y\rangle + |A_X Y|^2 - \langle Y, \nabla_X X\rangle^2 - \langle X, \nabla_Y Y\rangle^2$.

In this case, as F is minimal, [FF. III] becomes

$$\sum_{i=1}^{n-2} K(E_i, X) = \sum_i \langle \nabla_{E_i} \nabla_X X, E_i\rangle - \sum_i |T_{E_i} X|^2 - |A_X X|^2 - |A_X Y|^2 - 2\sum_i \langle \nabla_{E_i} X, Y\rangle \langle E_i, \nabla_X Y\rangle.$$

It follows that

$$\begin{aligned}
\mathrm{Ric}(X, X) &= \sum_{i=1}^{n-2} K(E_i, X) + K(X, Y) \\
&= \mathrm{div}(\nabla_X X) - \langle \nabla_X \nabla_X X, X\rangle + X\langle X, \nabla_Y Y\rangle - \sum_{i=1}^{n-2} |T_{E_i} X|^2 \\
&\quad - |A_X X|^2 - \langle Y, \nabla_X X\rangle^2 - \langle X, \nabla_Y Y\rangle^2 - 2\sum_i \langle \nabla_{E_i} X, Y\rangle \langle E_i, \nabla_X Y\rangle \\
&= \mathrm{div}(\nabla_X X) - \sum_{i=1}^{n-2} |T_{E_i} X|^2 + X\langle X, \nabla_Y Y\rangle - \langle X, \nabla_Y Y\rangle^2 \\
&\quad - 2\sum_i \langle \nabla_{E_i} X, Y\rangle \langle E_i, \nabla_X Y\rangle.
\end{aligned}$$

By the same way

$$\begin{aligned}\mathrm{Ric}(Y, Y) &= \sum_{i=1}^{n-2} K(E_i, Y) + K(Y, X) \\ &= \mathrm{div}(\nabla_Y Y) - \langle \nabla_Y \nabla_Y Y, Y\rangle + Y\langle Y, \nabla_X X\rangle - \sum_{i=1}^{n-2} |T_{E_i} Y|^2 \\ &\quad - |A_Y Y|^2 - \langle X, \nabla_Y Y\rangle^2 - \langle Y, \nabla_X X\rangle^2 - 2\sum_i \langle \nabla_{E_i} Y, X\rangle\langle E_i, \nabla_Y X\rangle \\ &= \mathrm{div}(\nabla_Y Y) - \sum_{i=1}^{n-2} |T_{E_i} Y|^2 + Y\langle Y, \nabla_X X\rangle - \langle Y, \nabla_X X\rangle^2 \\ &\quad - 2\sum_i \langle \nabla_{E_i} Y, X\rangle\langle E_i, \nabla_Y X\rangle.\end{aligned}$$

Thus we have

$$\begin{aligned}\mathrm{Ric}(X, X) + \mathrm{Ric}(Y, Y) &= \mathrm{div}(\nabla_X X + \nabla_Y Y) - \sum_{i=1}^{n-2} |T_{E_i} X|^2 - \sum_{i=1}^{n-2} |T_{E_i} Y|^2 \\ &\quad + X\langle X, \nabla_Y Y\rangle - \langle X, \nabla_Y Y\rangle^2 + Y\langle Y, \nabla_X X\rangle - \langle Y, \nabla_X X\rangle^2.\end{aligned}$$

As F is minimal, $\mathrm{div}(X) = \Sigma_i \langle \nabla_{E_i} X, E_i\rangle + \langle \nabla_X X, X\rangle + \langle \nabla_Y X, Y\rangle = -\langle \nabla_Y Y, X\rangle$ and $\mathrm{div}(Y) = -\langle \nabla_X X, Y\rangle$. It follows that $X\langle \nabla_Y Y, X\rangle - \langle \nabla_Y Y, X\rangle^2 = -X(\mathrm{div}(X)) - \mathrm{div}(X)^2 = -\mathrm{div}(\mathrm{div}(X)X)$ and that $Y\langle Y, \nabla_X X\rangle - \langle Y, \nabla_X X\rangle^2 = -\mathrm{div}(\mathrm{div}(Y)Y)$.

Therefore, we have

$$\mathrm{Ric}(X, X) + \mathrm{Ric}(Y, Y) + \sum_i |T_{E_i} X|^2 + \sum_i |T_{E_i} Y|^2 = \mathrm{div}(\nabla_X X + \nabla_Y Y - \mathrm{div}(X)X - \mathrm{div}(Y)Y).$$

By integrating this formula, we get

$$\int_M \{\mathrm{Ric}(X, X) + \mathrm{Ric}(Y, Y) + \sum_i |T_{E_i} X|^2 + \sum_i |T_{E_i} Y|^2\} = 0.$$

As each term of the integrand is non-negative, we have, in particular, $T_{E_i} X = T_{E_i} Y = 0$, that is, F is totally geodesic. This completes the proof.

Remark This result does not hold if $\mathrm{codim}(F) \geq 3$. In fact, let (S^3, g_0) be the three dimensional unit sphere. A suitable deformation of the Riemannian metric of $(S^3, g_0) \times (S^3, g_0)$ gives a new Riemannian manifold $(S^3 \times S^3, g)$ with non-negative Ricci curvature, so that $F = \{S^3 \times \{pt\}\}$ being minimal but non-totally geodesic and that $H = \{\{pt\} \times S^3\}$ being totally geodesic. As TS^3 is trivial, H is trivial.

As a corollary to this theorem, we have the following, which is a codimension-two version of Theorem 4.

Theorem 6 *Let (M, g, F) be a codimension-two foliation of a closed Riemannian manifold with non-negative Ricci curvature. Assume H is integrable and trivial. If the mean* curvature $\boldsymbol{H}$ of F is parallel along F, then $\boldsymbol{H} \equiv 0$. Consequently F is totally geodesic.

(*Proof.*) By Theorem 5, it suffices to show that $\boldsymbol{H} \equiv 0$. As $\nabla_E \boldsymbol{H} = 0$ for $E \in F$, we have $\langle \nabla_E E, \boldsymbol{H}\rangle = 0$ for $E \in F$. It follows that $0 = \Sigma_i \langle \nabla_{E_i} E_i, \boldsymbol{H}\rangle = |\boldsymbol{H}|^2$, which completes the proof.

Now return to the case of codimension q. Ler $\{E_i\}$ and $\{X_\alpha\}$ be as in [FF. III]. By summing up [FF. III], $\Sigma_i K(E_i, X_\alpha)$, for $\alpha=1, 2, \cdots, q$, we have

$$\text{[FF. IV]}\quad \sum_{i,\alpha} K(E_i, X_\alpha)$$
$$=\operatorname{div}(\boldsymbol{Q})+|\boldsymbol{Q}|^2+\sum_\alpha\langle\nabla_{X_\alpha}\boldsymbol{H}, X_\alpha\rangle-\sum_{i,j}|T_{E_i}E_j|^2-\sum_{\alpha,\beta}\langle A_{X_\beta}X_\alpha, A_{X_\alpha}X_\beta\rangle,$$

where $\boldsymbol{Q}=F(\Sigma_\alpha\nabla_{X_\alpha}X_\alpha)$. In fact,

$$\begin{aligned}\sum_{i,\alpha}K(E_i, X_\alpha)&=\sum_i\langle\nabla_{E_i}\boldsymbol{Q}, E_i\rangle+\sum_i\langle\nabla_{E_i}H(\sum_\alpha\nabla_{X_\alpha}X_\alpha), E_i\rangle+\sum_\alpha X_\alpha\langle X_\alpha, \boldsymbol{H}\rangle\\&\quad-\sum_{i,\alpha}|T_{E_i}X_\alpha|^2-\sum_{\alpha,\beta}\langle A_{X_\beta}X_\alpha, A_{X_\alpha}X_\beta\rangle\\&=\operatorname{div}(\boldsymbol{Q})-\sum_\alpha\langle\nabla_{X_\alpha}\boldsymbol{Q}, X_\alpha\rangle-\sum_i\langle H(\sum_\alpha\nabla_{X_\alpha}X_\alpha), \nabla_{E_i}E_i\rangle\\&\quad+\sum_\alpha X_\alpha\langle X_\alpha, \boldsymbol{H}\rangle-\sum_{i,j,\alpha}\langle T_{E_i}X_\alpha, E_j\rangle^2-\sum_{\alpha,\beta}\langle A_{X_\beta}X_\alpha, A_{X_\alpha}X_\beta\rangle\\&=\operatorname{div}(\boldsymbol{Q})+|\boldsymbol{Q}|^2-\sum_\alpha\langle\nabla_{X_\alpha}X_\alpha, \boldsymbol{H}\rangle+\sum_\alpha X_\alpha\langle X_\alpha, \boldsymbol{H}\rangle\\&\quad-\sum_{i,j}|T_{E_i}E_j|^2-\sum_{\alpha,\beta}\langle A_{X_\beta}X_\alpha, A_{X_\alpha}X_\beta\rangle.\end{aligned}$$

Theorem 7([O4]) *Let (M, g, F) be a codimension-q minimal foliation of a closed Riemannian manifold with non-negative Ricci curvature. If H is integrable and the induced normal connection of H is flat, then F is totally geodesic. Furthermore, (M, g) is locally a product of F and H.*

(*Proof.*) As the induced normal connection of H is flat, local orthonormal frame of H can be chosen so that $\langle\nabla_V X_\alpha, X_\beta\rangle=0$ for $V\in TM$. In order to show that F is totally geodesic, first calculate $K(X_\alpha, X_\beta)$.

$$\begin{aligned}K(X_\alpha, X_\beta)&=\langle\nabla_{X_\alpha}\nabla_{X_\beta}X_\beta, X_\alpha\rangle-\langle\nabla_{X_\beta}\nabla_{X_\alpha}X_\beta, X_\alpha\rangle-\langle\nabla_{[X_\alpha, X_\beta]}X_\beta, X_\alpha\rangle\\&=X_\alpha\langle\nabla_{X_\beta}X_\beta, X_\alpha\rangle-\langle\nabla_{X_\beta}X_\beta, \nabla_{X_\alpha}X_\alpha\rangle-X_\beta\langle\nabla_{X_\alpha}X_\beta, X_\alpha\rangle+\langle\nabla_{X_\alpha}X_\beta, \nabla_{X_\beta}X_\alpha\rangle\\&=|A_{X_\alpha}X_\beta|^2-\langle A_{X_\alpha}X_\alpha, A_{X_\beta}X_\beta\rangle\end{aligned}$$

As F is minimal and H is integrable, [FF. IV] becomes

$$\sum_{i,\alpha}K(E_i, X_\alpha)=\operatorname{div}(\boldsymbol{Q})+|\boldsymbol{Q}|^2-\sum_{i,j}|T_{E_i}E_j|^2-\sum_{\alpha,\beta}|A_{X_\beta}X_\alpha|^2.$$

Thus we have

$$\begin{aligned}\sum_\alpha\operatorname{Ric}(X_\alpha, X_\alpha)&=\sum_\alpha\{\sum_i K(E_i, X_\alpha)+\sum_{\beta\neq\alpha}K(X_\alpha, X_\beta)\}\\&=\operatorname{div}(\boldsymbol{Q})+|\boldsymbol{Q}|^2-\sum_{i,j}|T_{E_i}E_j|^2-\sum_{\alpha,\beta}\langle A_{X_\alpha}X_\alpha, A_{X_\beta}X_\beta\rangle\\&=\operatorname{div}(\boldsymbol{Q})-\sum_{i,j}|T_{E_i}E_j|^2,\end{aligned}$$

as $|\boldsymbol{Q}|^2=\langle\boldsymbol{Q}, \boldsymbol{Q}\rangle=\langle F(\Sigma_\alpha\nabla_{X_\alpha}X_\alpha), \Sigma_\beta\nabla_{X_\beta}X_\beta\rangle=\Sigma_{\alpha,\beta}\langle A_{X_\alpha}X_\alpha, A_{X_\beta}X_\beta\rangle$.

By integrating this formula, we have

$$\int_M\{\sum_\alpha\operatorname{Ric}(X_\alpha, X_\alpha)+\sum_{i,j}|T_{E_i}E_j|^2\}=0.$$

As each term of the integrand is non-negative, we have $\Sigma_{i,j}|T_{E_i}E_j|^2=0$, which means F is totally geodesic.

In order to prove that (M, g) is locally a product of F and H, we need a structure theorem of Riemannian manifolds with non-negative Ricci curvature given by Cheeger and Gromoll [CG] and foliation-preserving property of Killing vector fields given in Oshikiri [O3]. As our aim in this paper is mainly to present fundamental formulas of foliations and their applications to foliations, we omit the proof of local productness.

Though we do not give a proof of local productness here, we mention that the assumption of Theorem 5 implies the same conclusion as Theorem 7.

Theorem 8 *Let (M, g, F) be a codimension-two minimal foliation of a closed Riemannian manifold with non-negative Ricci curvature. If H is integrable and trivial, then (M, g) is locally a product of F and H.*

The first part of Theorem 7 can be extended as follows.

Theorem 9 ([Sw]) *Let (M, g, F) be a codimension-q minimal foliation of a closed Riemannian manifold with non-negative Ricci curvature. If H is integrable and the scalar curvature S_H of each leaf of H is non-positive, then F is totally geodesic.*

(*Proof.*) Let $X, Y \in H$ be unit vectors with $\langle X, Y\rangle=0$. By Gauss equation

$$K(X, Y)=K_H(X, Y)+\langle A_X Y, A_Y X\rangle-\langle A_X X, A_Y Y\rangle,$$

where $K_H(X, Y)$ is the sectional curvature of the plane spanned by X and Y in a leaf of H with the induced metric. It follows that

$$\begin{aligned}\sum_\alpha \mathrm{Ric}(X_\alpha, X_\alpha) &= \sum_\alpha\{\sum_i K(E_i, X_\alpha)+\sum_{\beta\neq\alpha} K(X_\alpha, X_\beta)\}\\ &=\sum_{i,\alpha} K(E_i, X_\alpha)+\sum_{\alpha,\beta}\{K_H(X_\alpha, X_\beta)+\langle A_{X_\alpha}X_\beta, A_{X_\beta}X_\alpha\rangle-\langle A_{X_\alpha}X_\alpha, A_{X_\beta}X_\beta\rangle\}\\ &=\sum_{i,\alpha} K(E_i, X_\alpha)+S_H+\sum_{\alpha,\beta}|A_{X_\alpha}X_\beta|^2-|\boldsymbol{Q}|^2.\end{aligned}$$

As F is minimal and H is integrable, [FF. IV] becomes

$$\sum_{i,\alpha} K(E_i, X_\alpha)=\mathrm{div}(\boldsymbol{Q})+|\boldsymbol{Q}|^2-\sum_{i,j}|T_{E_i}E_j|^2-\sum_{\alpha,\beta}|A_{X_\alpha}X_\beta|^2.$$

It follows that

$$\sum_\alpha \mathrm{Ric}(X_\alpha, X_\alpha)=\mathrm{div}(\boldsymbol{Q})+S_H-\sum_{i,j}|T_{E_i}E_j|^2.$$

Thus, by integrating this formula, we have

$$\int_M\{\sum_\alpha \mathrm{Ric}(X_\alpha, X_\alpha)+(-S_H)+\sum_{i,j}|T_{E_i}E_j|^2\}=0.$$

As each term of the integrand is non-negative, we have $\Sigma_{i,j}|T_{E_i}E_j|^2=0$, which means F is totally geodesic. This completes the proof.

On the lines of O'Neill's paper [ON], Ranjan [Rj] derived the structural equations for foliations and derived some global results. As the results are essentially contained in our context, we simply present his main results.

Theorem 10([Rj]) *For foliated closed manifold, we have*

$$\int_M \sum_{i,\alpha} K(E_i, X_\alpha)=\int_M\{|\boldsymbol{H}|^2-\sum_{i,j}|T_{E_i}E_j|^2\}+\int_M\{|\boldsymbol{Q}|^2-\sum_\alpha \mathrm{Tr}(A^2_{X_\alpha})\}$$

Theorem 11([Rj]) *Let M be a closed manifold. Let V be a unit tangent vector field on M which gives a metric foliation on M, then $\int_M \mathrm{Ric}(V, V)\geq 0$ with equality iff the complementary codimension-one subbubdle is integrable.*

Theorem 12([Rj]) *Let (M, g, F) be a codimension-one minimal foliation. Then $\int_M \mathrm{Ric}(N, N)\leq 0$ with equality iff F is totally geodesic.*

5. Complete case

In this section, we try to apply our formulas for non-compact cases. There are only a few results on differential geometric properties of foliations of non-compact manifolds (cf. [BGG], [Me], [Rh]).

Theorem 13([Ab]) *Let (M, g, F) be a codimension-q totally geodesic foliation of a complete Riemannian manifold with non-negative sectional curvature. If H is integrable, then H is also totally geodesic, and (M, g) is locally a product of F and H.*

(*Proof.*) Let γ be a geodesic along F with $\gamma(0)=x\in M$. Take an orthonormal basis $\{e_1, e_2, \cdots, e_q\}$ of H_x and let $\{X_1, X_2,\cdots, X_q\}$ be the orthonormal frame field along γ obtained by the parallel translations of $\{e_1, e_2, \cdots, e_q\}$. As F is totally geodesic, $\{X_1, X_2, \cdots, X_q\}$ is an orthonormal frame field of H along γ. By summing up $\Sigma_{\alpha=1}^q \mathrm{K}(\dot\gamma, X_\alpha)$, [FF. I] gives

$$\sum_\alpha K(\dot\gamma, X_\alpha)=\dot\gamma\langle \boldsymbol{Q}, \dot\gamma\rangle-\sum_{\alpha,\beta}\langle A_{X_\alpha}X_\beta, \dot\gamma\rangle^2$$

because $\nabla_{\dot\gamma}\dot\gamma=0$ and $\nabla_{\dot\gamma}X_\alpha=0$. It follows that

$$\dot\gamma\langle Q, \dot\gamma\rangle-\frac{\langle \boldsymbol{Q}, \dot\gamma\rangle^2}{q}\geq\dot\gamma\langle \boldsymbol{Q}, \dot\gamma\rangle-\sum_\alpha\langle A_{X_\alpha}X_\alpha, \dot\gamma\rangle^2$$
$$\geq\sum_{\alpha\neq\beta}\langle A_{X_\alpha}X_\beta, \dot\gamma\rangle^2+\sum_\alpha K(\dot\gamma, X_\alpha).$$

Set $f(t)=\langle \boldsymbol{Q}, \dot{\gamma}(t)\rangle$. We may assume $f(0)\geq 0$. Then, as the sectional curvature is non-negative, we have

$$f'(t)-\frac{f(t)^2}{q}\geq 0 \quad \text{for } t\in \boldsymbol{R}.$$

Now consider the differential equation

$$y'=\frac{y^2}{q}.$$

If $y(t)$ is the solution of this differential equation with $y(0)=f(0)$, then $f(t)\geq y(t)$ for $t\geq 0$. The solution $y(t)$ of $y'=y^2/q$ with $y(0)=c$ is given by

$$y(t)=\frac{cq}{q-ct}.$$

If $c>0$, then $y(t)\to\infty$ as $t\uparrow q/c$. As $f(t)$ is defined for all $t\in\boldsymbol{R}$, this is a contradiction. Thus $c=0$. If $\boldsymbol{Q}\neq 0$ at $x\in M$, then we can choose $\gamma(t)$ so that $f(0)=\langle \boldsymbol{Q}, \dot{\gamma}(0)\rangle>0$. This also gives a contradiction. Therefore, we have $\boldsymbol{Q}\equiv 0$ and

$$\sum_{\alpha} K(\dot{\gamma}, X_\alpha)=-\sum_{\alpha,\beta}\langle A_{X_\alpha}X_\beta, \dot{\gamma}\rangle^2$$

As $K(\dot{\gamma}, X_\alpha)\geq 0$, it follows that $A_{X_\alpha}X_\beta=0$, which means H is totally geodesic. This completes the proof.

Theorem 14 *Let (M, g, F) be a codimension-one minimal foliation of a complete Riemannian manifold with non-negative Ricci curvature. If the growth* $\mathrm{gr}(L)$ *of a leaf $L\in F$ is not greater than 2, then L is totally geodesic. In particular, if* $\mathrm{gr}(F)\leq 2$, *then F is totally geodesic, and (M, g) is locally a product of F and H.*

(Proof.) Let L be a leaf of F. By assumption, [FF. II] becomes

$$\mathrm{Ric}(N, N)+\mathrm{Tr}(T_N^2)+|\theta|^2=\mathrm{div}_L(\nabla_N N).$$

If L is compact, by integrating this formula over L, we have

$$\int_L \{\mathrm{Ric}(N, N)+\mathrm{Tr}(T_N^2)+|\theta|^2\}=0.$$

As each term of the integrand is non-negative, it follows that $\mathrm{Tr}(T_N^2)=0$, which means L is totally geodesic.

Now assume L is a non-compact leaf with the growth $\mathrm{gr}(L)\leq 2$. Fix $x\in L$. Then, by definition,

$$\mathrm{vol}(B(r))\leq ar^2+b \quad (r\geq 0),$$

for some positive constants a and b, where $B(r)=\{y\in L\,|\,d_L(x, y)\leq r\}$. Set $f(r)=\int_{B(r)}|\theta|^2$ and $V(r)=\mathrm{vol}(B(r))$, where θ is the dual one-form of $\nabla_N N$. It is known that $f(r)$ and $V(r)$ are locally Lipschitz, and thus a. e. differentiable. By integrating

$$\mathrm{Ric}(N, N)+\mathrm{Tr}(T_N^2)+|\theta|^2=\mathrm{div}_L(\nabla_N N)$$

over $B(r)$, we have

$$\int_{B(r)}\{\mathrm{Ric}(N, N)+\mathrm{Tr}(T_N^2)+|\theta|^2\}=\int_{B(r)}\mathrm{div}(\nabla_N N)=\int_{\partial B(r)}\theta(v)$$

where v is the outward unit normal vector to $\partial B(r)$. As $\mathrm{Ric}(N, N)\geq 0$ and $\mathrm{Tr}(T_N^2)\geq 0$, we have

$$\int_{B(r)}|\theta|^2\leq\int_{\partial B(r)}|\theta|\leq\sqrt{\int_{\partial B(r)}1}\sqrt{\int_{\partial B(r)}|\theta|^2}.$$

It follows that

$$f(r)^2\leq f'(r)V'(r),$$

because $f'(r)=\int_{\partial B(r)}|\theta|^2$ and $V'(r)=\int_{\partial B(r)}1$.

Assume $\theta(x)\neq 0$. Then $f(r)>0$ for $r>0$. As $V'(r)>0$, we have

$$\frac{1}{V'(r)}\leq\frac{f'(r)}{f(r)^2}=\left(-\frac{1}{f(r)}\right)'.$$

Integrating this on $[r, R]$ with $0<r<R$, we get

$$\int_r^R\frac{1}{V'(r)}dr\leq\frac{1}{f(r)}-\frac{1}{f(R)}.$$

The inequality

$$\left(\int_r^R dr\right)^2=\left(\int_r^R\sqrt{V'(r)}\sqrt{\frac{1}{V'(r)}}dr\right)^2\leq\left(\int_r^R V'(r)dr\right)\left(\int_r^R\frac{1}{V'(r)}dr\right)$$

implies

$$\frac{(R-r)^2}{V(R)-V(r)}\leq\int_r^R\frac{1}{V'(r)}dr.$$

It follows that

$$\frac{(R-r)^2}{V(R)-V(r)}\leq\frac{1}{f(r)}-\frac{1}{f(R)}.$$

Letting $R=2r$, we have

$$\frac{r^2}{4ar^2+b}\leq\frac{r^2}{V(2r)}\leq\frac{r^2}{V(2r)-V(r)}\leq\frac{1}{f(r)}-\frac{1}{f(2r)}.$$

As $f'(r)\geq 0$, if $f(r)$ is bounded above, then the above inequality implies

$$0<\frac{1}{8a}\leq\frac{r^2}{4ar^2+b}\leq\frac{1}{f(r)}-\frac{1}{f(2r)}\to 0\ (\text{as } r\to\infty),$$

which is a contradiction. If $f(r)$ tends to the infinity as $r\to\infty$, then we have

$$0<\frac{1}{8a}\leq\frac{r^2}{4ar^2+b}\leq\frac{1}{f(r)}-\frac{1}{f(2r)}\to 0\ (\text{as } r\to\infty),$$

which is also a contradiction. Therefore we have $f(r)\equiv 0$, that is $\nabla_N N\equiv 0$ on L. This also implies L is totally geodesic.

If $\mathrm{gr}(F)=\max\ \{\mathrm{gr}(L)\,|\,L\in F\}\leq 2$, then, by the above argument, F is totally geodesic and $\nabla_N N\equiv 0$ on M. Thus (M, g) is locally a product of F and H. This completes the proof.

Finally, we see that the formula [FF. II] can be used to show the stability of some leaves of codimension-one constant mean curvature foliations, and leaves of codimension-one minimal foliations.

Theorem 15 (cf. [O2]) *Let (M, g, F) be a codimension-one constant mean curvature foliation of a complete Riemannian manifold. If $N(h)\geq 0$, where $h=\langle \boldsymbol{H}, N\rangle$, on $L\in F$, then L is stable. Inparticular, every leaf of a codimension-one minimal foliation is stable.*

(*Proof.*) Note that, if h is constant on each leaf of F, then either $N(h)\equiv 0$ or $N(h)\neq 0$ on $L\in F$. Thus, if $N(h)\neq 0$ at a point $x\in L$, then $N(h)>0$ or $N(h)<0$ on L.

A constant mean curvature leaf $L\in F$ with $h\neq 0$ is said to be stable if

$$V_f''(0)=\int_L\{|df|^2-f^2(\mathrm{Ric}(N, N)+\mathrm{Tr}(T_N^2))\}\geq 0,$$

where f is any function having compact support on L with $\int_L f=0$. For a minimal leaf L, the definition of the stability of L is given by the same way as above without the condition $\int_L f=0$ for f.

By [FF. II], it follows that

$$\begin{aligned}f^2(\mathrm{Ric}(N, N)+\mathrm{Tr}(T_N^2))&=f^2\mathrm{div}(\nabla_N N)-f^2N(h)\\&=f^2\mathrm{div}_L(\nabla_N N)-f^2|\theta|^2-f^2N(h)\\&=-\nabla_N N(f^2)-f^2|\theta|^2-f^2N(h)+\mathrm{div}_L(f^2\nabla_N N).\end{aligned}$$

Thus, we have

$$\begin{aligned}V_f''(0)&=\int_L\{|df|^2+2f\nabla_N N(f)+f^2|\theta|^2+f^2N(h)\}\\&=\int_L|df+f\theta|^2+\int_L f^2N(h).\end{aligned}$$

If $N(h)\geq 0$ on L, then $V''_f(0)\geq 0$, that is, L is stable.

Note that the condition $\int_L f=0$ is not used here. Thus, if F is minimal, then, as $N(h)\equiv 0$, $V''_f(0)=\int_L|df+f\theta|^2\geq 0$, that is, L is stable.

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