

On the Propagation of Shock Waves

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(Received July 6, 1978)

Abstract

For a strong shock wave propagating into an expanding channel, an improved characteristic method is re-explained in a slightly different manner. A new physical quantity (labelled χ) is introduced for the method. The quantity indicates a magnitude of damping ($\chi > 0$) or growing ($\chi < 0$) of the shock wave. Then, the positive (minor) characteristic equation is converted to the major characteristic one along the particle path line. By simultaneous using of this characteristic equation, the negative characteristic one and Brinkley-Kirkwood's (1947) method for the shock propagation, the entire flow field is determined as a function of χ . The solutions are in good agreement with similarity ones. Accordingly, this method is applied to the propagation of cylindrical shock waves under the actions of the differential rotation of gas itself, the gravitational field of the other mass and the circular magnetic field, and the governing equations are formulated. Furthermore, various methods are briefly outlined and compared with the above method. Especially, for Chisnell's method it is found that the propagation of a shock wave can be solved analytically under the action of gravity. The solution brings out the clear perspective of the variation of the shock strength.

1. Introduction

Simple waves in a flow field frequently grow to shock waves when nonlinearity and dissipation terms (e. g., viscosity and thermal conduction) are important in governing equations. The shock waves propagate into the medium and affect on the flow field. Thus the shock propagation should be determined together with the flow field from the governing equations and the shock-jump relations. However, even in a uniform medium, the problem of the shock propagation cannot be solved analytically in general.

In recent years, to solve various problems arising in the different branches of fluid dynamics, numerical calculation techniques have been largely developed (Boris 1976). It is also an important subject to clarify the physical properties of the shock propagation. This problem has been investigated by many authors from different points of view with various methods.

Among them the characteristic method has the advantage of following points, although dissipation terms are not included:

- (i) In determining the flow field behind the shock front and shock propagation, the

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role of disturbances generated by any element in the fluid is taken into account ;

- (ii) By transforming the partial differential equations into ordinary ones, the Riemann invariants of the flow field are introduced with a distinct physical meaning.

For this method, Whitham (1958) has suggested a simple rule to determine the shock propagation in a non-uniform tube. This rule is to apply the differential equation which is valid along a minor characteristic to the flow quantities just behind the shock front. Furthermore, he has suggested that the shock-expansion method is related to Chisnell's method, where the reflection and penetration of the shock wave are taken into account at positions of infinitesimal density jumps, i. e., a small perturbation method (Chisnell 1957 ; Ôno, Sakashita and Yamazaki 1960). Thereafter Lick (1966) has analyzed a similar problem, using the shock-expansion method, and pointed out a way of calculating the entire flow field for the adiabatic motion. His formulation reduces to Whitham's rule at the shock front.

Whitham's rule is sufficient to determine only the propagation of the shock wave for a restricted class of problems. That is, the rule is invalid for a strong shock wave propagating into an expanding channel where the remaining effect of reflected waves is appeared. The reflected waves bring out damping or growing of the shock wave. This means that the propagation of the shock wave should be determined together with the flow field behind the shock front.

The interplay of the shock wave and the flow field gives rise to difficulties in solving the same problem. In order to overcome the difficulties, we have proposed an improvement on the characteristic method by introducing a new quantity with reference to Whitham's rule (Saitô, M. and Saito, Y. 1977). This quantity denotes a magnitude of damping for a positive value or growing for a negative one. Then, by simultaneous using of the above quantity, the characteristic equations and Brinkley-Kirkwood's (1947) method, the radial propagation of a strong shock wave in an inhomogeneous medium has been solved under the actions of rotation, magnetic and gravitational forces.

In this paper, our method is re-examined in a slightly different manner and the meanings of the quantity are clarified as follows. By introducing the quantity, the positive (minor) characteristic equation is converted to the major characteristic one along the particle path line. Then the behavior of the basic equations (two kinds of the major characteristic equations and the propagation equation of the shock wave) can be analyzed for various values of the quantity as a parameter. If the quantity is equal to zero, the basic equations reduce to the basic ones in the other methods. Especially, the shock-expansion method reduces to Whitham's rule just behind the shock front. For a strong cylindrical shock, the values and solutions are determined simultaneously. The results are in good agreement with those of the similarity solution (Sedov 1959 ; Gross 1971).

In section 2, the various methods are outlined and some critical discussions for them are given. Particularly, for Chisnell's method, it is found that the propagation of the shock wave can be solved analytically under the action of gravity. The solution brings out the

clear perspective of the shock strength. In section 3, our method is re-explained in a slightly different manner to provide facilities for comparison with the other methods. In section 4, our results are discussed in comparison with those of the other method.

2. The Methods of Shock Propagation

In this section, the different methods, i. e., the methods of quasi-stationary propagation (Chisnell 1957), characteristic (Whitham 1958), expansion (Lick 1966) and similarity solution (Sedov 1959) are outlined. Other numerical solution techniques are not referred for the reasons mentioned already.

2-1. Chisnell's Method

Chisnell (1957) has considered a shock wave propagating down a channel of variable area with inhomogeneous conditions for the gas which is at rest ahead of the shock front. He has divided an inhomogeneous fluid produced by the shock wave into layers with infinitesimal density jumps and constant pressure, and considered the reflection and penetration of the shock wave at these jump positions. Consequently, he has obtained an ordinary differential equation of the shock propagation and integrated the equation.

If $S(r)$ is constant in $r < r_0$ and the shock wave initially moving with velocity U in this region, the ordinary differential equation is written as

$$-\frac{1}{S} \frac{dS}{dz} = \frac{1-\nu^2}{(1+\nu^2)z} + \frac{2}{z-1} - \frac{1}{2(z+\nu^2)} - \frac{1-\nu^2}{\sqrt{(1+\nu^2)z(\nu^2z+1)}} \left\{ 1 - \frac{z-1}{2(z+\nu^2)} + \frac{\nu^2z+1}{(1-\nu^2)(z-1)} \right\} \quad (1)$$

with $\nu = \sqrt{(\gamma-1)/(\gamma+1)}$, where $S(r)$ is the variable cross-section area, $z (= P_{g2}/P_{g1})$ the shock strength, γ the ratio of the specific heat at constant pressure C_{Pg} to that at constant volume C_v and r (or x) the spatial coordinate. Shock waves are classified by the cross-sectional areas $S(r)$ for $r \geq r_0$. Then the plane parallel, cylindrical and spherical shocks are obtained by choosing a uniform tube $S(r) = \text{constant}$, wedge-shaped one $S(r) \propto r$ and cone-shaped one $S(r) \propto r^2$, respectively. The shock-jump relations which are called as the Rankine-Hugoniot relations are,

$$P_{g2} - P_{g1} = \rho_1 u_2 U, \quad \rho_1 U = \rho_2 (U - u_2) \quad \text{and} \quad w_2 + \frac{1}{2}(U - u_2)^2 = w_1 + \frac{1}{2}U^2, \quad (2)$$

where P_g is the gas pressure, ρ the density, u the velocity in the r (or x)-direction and w the specific enthalpy. Hereafter, the subscript "1" specifies the unperturbed state ahead of the shock front and the subscript "2" the state just behind the shock front. The equation of state is $P_g = \rho R T$, and then the isentropic speed of sound is defined as $c_s^2 \equiv (\partial P_g / \partial \rho)_\phi = \gamma P_g / \rho$, where R is the gas constant, T the temperature and ϕ the specific entropy.

Ôno, Sakashita and Yamazaki (1960) have generalized Chisnell's method under the

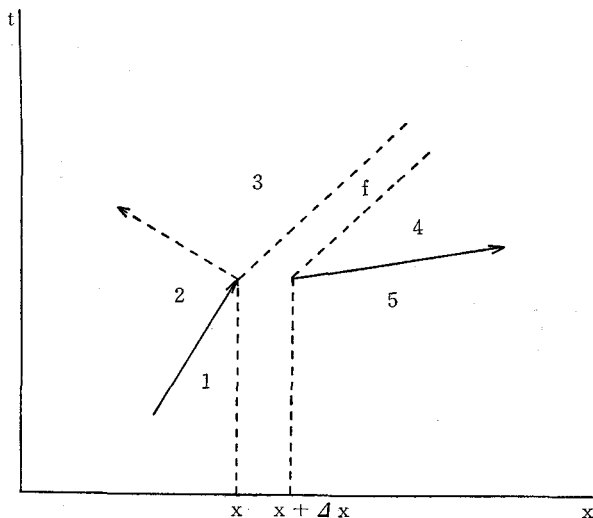


Fig. 1. An interaction of a shock front with an infinitesimal layer. $\langle 1, 2 \rangle$ denotes the shock jump, $\langle 2, 3 \rangle$ the reflected wave, $\langle 3, 4 \rangle$ the discontinuity surface, $\langle 4, 5 \rangle$ the penetrated shock wave, $\langle 1, 5 \rangle$ the initial infinitesimal pressure and density jumps, and "f" a local force. (Öno, et al 1961)

action of gravity, where pressure as well as density, and applied their results to the problem of the propagation of a shock wave which has been generated in some way in a stellar interior (see figure 1.). Assuming the initial distribution of the polytropic gas, i. e., $P_{g1} \propto \rho_1^{\gamma'}$, they have obtained

$$\frac{1}{P_{g1}} \frac{dP_{g1}}{dz} = \frac{\gamma'}{\rho_1} \frac{d\rho_1}{dz} = \frac{\frac{1}{\nu^2+z} - \frac{2}{z-1} - \frac{2}{z-1} \left\{ \frac{1+\nu^2 z}{(1+\nu^2)z} \right\}^{\frac{1}{2}}}{2 \left\{ \frac{1+\nu^2 z}{(1+\nu^2)z} \right\}^{\frac{1}{2}} + \frac{\gamma'-1}{\gamma'}} \quad (3)$$

where γ' is the polytropic exponent. Chisnell's case corresponds to $\gamma'=0$ (isobar). Then equation (1) is re-written as

$$\frac{1}{\rho_1} \frac{d\rho_1}{dz} = \frac{2}{z-1} - \frac{1}{\nu^2+z} + \frac{2}{z-1} \left\{ \frac{1+\nu^2 z}{(1+\nu^2)z} \right\}^{\frac{1}{2}}. \quad (1')$$

Both equations (1) and (1)' can be integrated analytically. They have solved equation (3) numerically except for an isothermal case ($\gamma'=1$), but it can be integrated analytically for arbitrary values of γ' . To do so, in this equation, we choose its denominator as a new variable instead of z and rationalize the integrand. After a few manipulations, we obtain

$$\rho_1 = Af(z), \quad (4)$$

$$\{f(z)\}^{\gamma'} = z^{\alpha_1} (\nu^2+z)^{\alpha_2} \left\{ 2 \sqrt{\frac{1+\nu^2 z}{(1+\nu^2)z}} + \frac{\gamma'-1}{\gamma'} \right\}^{\alpha_3}$$

$$\begin{aligned} & \times \left\{ \frac{1}{\nu} - \sqrt{\frac{z}{1+\nu^2 z}} \right\}^{\alpha_4} \left[\frac{\sqrt{(1+\nu^2)z} + 1}{(z-1)\sqrt{\frac{(1+\nu^2)z}{1+\nu^2 z} - 1}} \right]^{\alpha_5} \\ & \times \exp\left\{ \alpha_6 \tan^{-1} \frac{1}{\nu} \sqrt{\frac{(1-\nu^2)(1+\nu^2)z}{1+\nu^2 z}} \right\}, \end{aligned} \quad (5)$$

where A is an integral constant which is so determined that at pressure P_{g0} the shock strength is z_0 .

$$\begin{aligned} \alpha_1 &= (\gamma' - 1) \left[\frac{1}{4\gamma'} \left\{ \frac{1+\nu^2}{1 - \left(\frac{\gamma'-1}{2\gamma'}\right)^2 \frac{1+\nu^2}{\nu^2}} - \frac{\frac{\nu^2}{1-\nu^2}}{1 + \left(\frac{\gamma'-1}{2\gamma'}\right)^2 \frac{\nu^2}{1-\nu^2}} \right\} - \frac{1}{3\gamma'-1} \right], \\ \alpha_2 &= \frac{\gamma'-1}{4\gamma'} \frac{\frac{\nu^2}{1-\nu^2}}{1 + \left(\frac{\gamma'-1}{2\gamma'}\right)^2 \frac{\nu^2}{1-\nu^2}}, \\ \alpha_3 &= \frac{(\gamma'-1)(\gamma'+1)}{2\gamma'^2(3\gamma'-1)} \cdot \frac{\gamma' - (3\gamma'-1)\nu^2}{\nu^2(1-\nu^2) \left\{ 1 - \left(\frac{\gamma'-1}{2\gamma'}\right)^2 \frac{1+\nu^2}{\nu^2} \right\} \left\{ 1 + \left(\frac{\gamma'-1}{2\gamma'}\right)^2 \frac{\nu^2}{1-\nu^2} \right\}}, \\ \alpha_4 &= \frac{\sqrt{1+\nu^2}}{2\gamma'\nu} \cdot \frac{1}{1 - \left(\frac{\gamma'-1}{2\gamma'}\right)^2 \frac{1+\nu^2}{\nu^2}}, \\ \alpha_5 &= \frac{2\gamma'}{3\gamma'-1}, \\ \alpha_6 &= \frac{\nu}{\sqrt{1-\nu^2}} \cdot \frac{1}{1 + \left(\frac{\gamma'-1}{2\gamma'}\right)^2 \frac{\nu^2}{1-\nu^2}}. \end{aligned} \quad (6)$$

For a strong and weak shock waves, the leading term of $f(z)$ are proportional to $(z-1)^{-2\alpha_6}$ and z^{α_7} , respectively, where $\alpha_7 = -\{1 + \sqrt{1+\nu^2}/(2\nu)\} / \{1 + (\gamma'-1)\sqrt{1+\nu^2}/(2\gamma'\nu)\}$.

For $\gamma'=1$ (isothermal), we obtain

$$\rho_1(z) = \frac{A}{z-1} \cdot \left(\frac{1 - \sqrt{\frac{z}{1+\nu^2 z}}}{\frac{1}{\nu} + \sqrt{\frac{z}{1+\nu^2 z}}} \right)^{\frac{\sqrt{1+\nu^2}}{2\nu}} \cdot \frac{\sqrt{\frac{(1+\nu^2)z}{1+\nu^2 z}} + 1}{\sqrt{\frac{(1+\nu^2)z}{1+\nu^2 z}} - 1} \cdot \exp\left\{ -\frac{\nu}{\sqrt{1-\nu^2}} \tan^{-1} \frac{1}{\nu} \sqrt{\frac{(1-\nu^2)(1+\nu^2)z}{1+\nu^2 z}} \right\}. \quad (7)$$

(There is a misprint in Ôno, Sakashita and Yamazaki's (3.9); this function should read the function (7)).

Substituting the polytropic relation in (4), we obtain the relation between the shock strength z and initial pressure P_{g1} ;

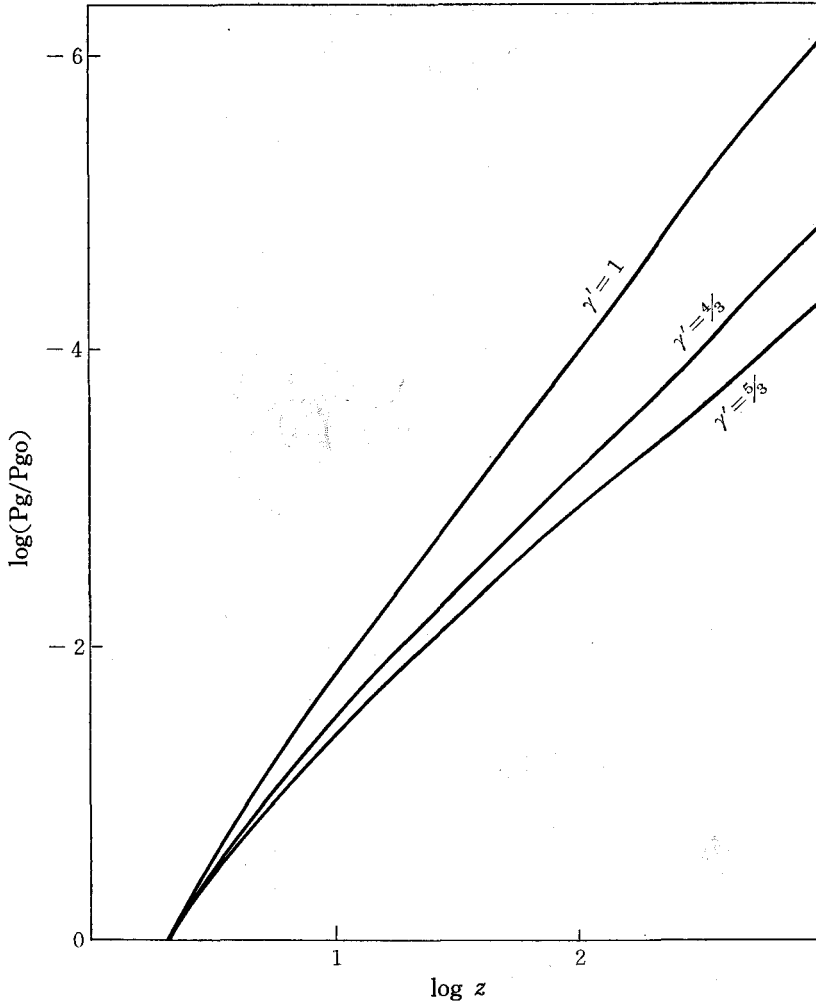


Fig. 2. Shock strength z versus initial gas pressure P_g/P_{g0} .

$$\log \frac{P_{g1}}{P_{g0}} = \gamma' \log \left\{ \frac{f(z)}{f(z_0)} \right\}, \quad (8)$$

In figure 2, relation (8) is drawn for the following case $\gamma' = 1$ (isothermal), $4/3$ (corresponding to the Eddington model) and $\gamma' = 5/3$ (adiabatic) with the ratio of specific heats $\gamma = 5/3$ (the value of one atomic gas which corresponds to stellar interior) and $z_0 = 2.2$ ($f(z_0) = 0.704$). We see that the shock strength increases rapidly as pressure decreases from the center to the surface in the stellar interior, but the degree of increment differ among differnt values of γ' ; it is the least for the isothermal distribution and increases as γ' is large. As the results of the increasing strength from the starting points to the surface, the other physical quantities, i. e., density, temperture and gaseous velocity, also increase considerably behind the shock front. Ôno, Sakashita and Yamazaki

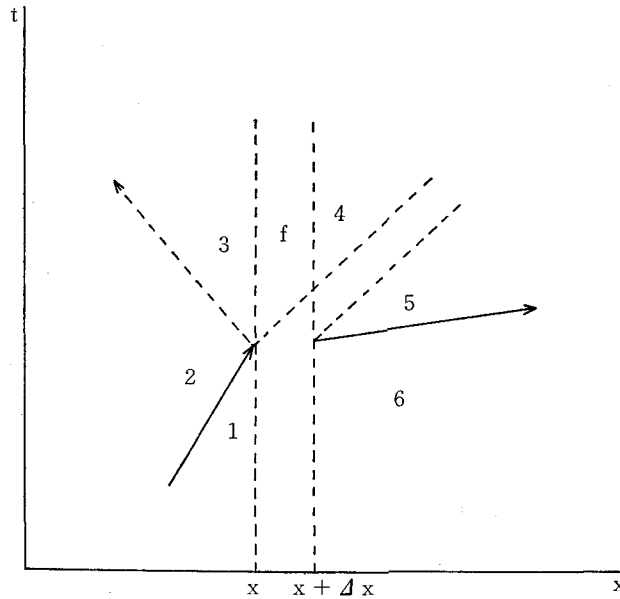


Fig. 3. An interaction of a shock front with an infinitesimal layer. The notations are the same with figure 1. (Ono et al)

has applied their results to the Eddington model and made some speculations concerning the origin of nova explosion.

2-2. Whitham's Rule

Nextly, Whitham (1958) has studied the same problem in the preceding sub-section, based on the method of the characteristic. He has dealt with this problem as follows.

When the shock wave reaches $r=r_0$, disturbances are propagated back into the uniform flow region and the future motion of the shock wave is modified. This reflected disturbances propagate along negative characteristics labelled C^- . In addition, entropy changes are carried along the particle paths labelled C^0 . From the physical point of view, the positive characteristics labelled C^+ play a subsidiary role which is seen later.

The equations of continuity, motion and entropy for the one-dimensional, time (t)-dependent flow of an inviscid, non conducting perfect gas are

$$\frac{D\rho}{Dt} + \rho \frac{\partial u}{\partial r} + \frac{\rho u}{S(r)} \frac{\partial S(r)}{\partial r} = 0, \quad (9)$$

$$\frac{Du}{Dt} = -\frac{1}{\rho} \frac{\partial P_g}{\partial r}, \quad (10)$$

$$\frac{D\phi}{Dt} = 0, \quad (11)$$

with

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial r}. \quad (12)$$

Then the characteristic equations can be written as

$$dP_g + \rho c_\phi du + \frac{\rho c_\phi^2 u}{u + c_\phi} \cdot \frac{dS}{S} = 0 \text{ on } C^+, \quad \frac{dr}{dt} = u + c_\phi; \quad (13)$$

$$dP_g - \rho c_\phi du + \frac{\rho c_\phi^2 u}{u - c_\phi} \cdot \frac{dS}{S} = 0 \text{ on } C^-, \quad \frac{dr}{dt} = u - c_\phi; \quad (14)$$

$$dP_g - c_\phi^2 d\rho = 0 \text{ on } C^0, \quad \frac{dr}{dt} = u. \quad (15)$$

Substituting the Rankine-Hugoniot relations (2) without the subscript "2" and the expression of the Mach number $M(=U/c_\phi)$ in equation (13), Whitham has obtained a differential equation for $M(S)$ which is identical with Chisnell's equation (1). This means that the positive C^+ characteristic equation are applicable to the shock front. The justification of this procedure is as follows:

$$\sigma^+ \equiv \left(\frac{1}{U} \frac{\partial P_g}{\partial t} + \frac{\partial P_g}{\partial r} \right) + \rho c_\phi \left(\frac{1}{U} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial r} \right) + \frac{\rho c_\phi^2 u}{S(u + c_\phi)} \frac{dS}{dr} \quad (16)$$

must be nearly equal to zero. Subtracting the positive characteristic equation (13),

$$0 = \left(\frac{1}{u + c_\phi} \frac{\partial P_g}{\partial t} + \frac{\partial P_g}{\partial r} \right) + \rho c_\phi \left(\frac{1}{u + c_\phi} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial r} \right) + \frac{\rho c_\phi^2 u}{S(u + c_\phi)} \cdot \frac{dS}{dr}, \quad (13)'$$

We obtain the following expression;

$$\sigma^+ = \left(\frac{1}{U} - \frac{1}{u + c_\phi} \right) \left(\frac{\partial P_g}{\partial t} + \rho c_\phi \frac{\partial u}{\partial t} \right). \quad (17)$$

The first factor in the above expression, i. e., $(u + c_\phi - U)/U$, is zero for $M=1$ and then $\sigma^+=0$ corresponds to Lick's formulation (24) (as seen later) at the shock front. On the other hand, following the small perturbation theory (Chester 1954), the second factor is very small along each positive characteristic. Therefore σ^+ is nearly equal to zero. This means that the positive characteristic equation (13) can be applied to the flow quantities just behind the shock front and so is the minor characteristic one.

Whitham's rule is applicable to a wide range of problems, but the accuracy depends on the cancellation of reflected waves. For instance, his rule is invalid for a strong shock wave propagating into an expanding channel where the remaining effect of reflected waves is appeared. Thus the propagation of this shock should be determined together with the flow field behind the shock front.

2-3. The Method of Shock-Expansion

Lick (1966) has considered the time-dependent, one dimensional motion of a piston accelerated from rest with non-zero initial velocity, as an example of the application of the shock-expansion method. Then a shock wave is formed instantaneously, because the gas ahead of the shock front is motionless and in a uniform state. The shock propagation is expressed in terms of only one space coordinate, and is in a non-uniform state of neither

spherical nor cylindrical symmetry, but plane parallel. Thus, in equation (9), $S(x)$ is constant in $x \geq x_0$ and then $dS(x)$ becomes zero. As is well known, this means that there is not the geometrical damping. Disturbances originating from the piston surface and due to the variable motion of the piston propagate along the positive C^+ characteristics. These disturbances are partly absorbed by the shock wave with the variable speed and partly reflected. The reflected waves propagate along the negative C^- characteristics and eventually influence the pressure on the piston surface.

Replacing dS by zero, we can re-write equations (13), (14) and (15) as

$$dG - \frac{(1-\nu^2)c_\phi}{4\nu^2 C_{P_g}} d\phi = 0 \text{ on } C^+, \quad \frac{dx}{dt} = u + c_\phi; \quad (18)$$

$$dH - \frac{(1-\nu^2)c_\phi}{4\nu^2 C_{P_g}} d\phi = 0 \text{ on } C^-, \quad \frac{dx}{dt} = u - c_\phi; \quad (19)$$

$$d\phi = 0 \quad \text{on } C^0, \quad \frac{dx}{dt} = u \quad (11)'$$

with convenient transformations

$$G = \frac{1}{2} \left(\frac{1-\nu^2}{\nu^2} c_\phi + u \right), \quad (20)$$

$$H = \frac{1}{2} \left(\frac{1-\nu^2}{\nu^2} c_\phi - u \right). \quad (21)$$

Then the variable G and H introduced above reduce to the Riemann invariants (labelled J^+ and J^- , respectively), provided that $dP_g/(\rho c_\phi)$ is the total differential. However, since the entropy of the gas is increased across the shock front, these reductions do not occur. If the shock wave is weak, it is approximately a simple wave. Then J^+ is constant on the positive characteristic and J^- constant on the negative one.

As mentioned already, the disturbances generated in the accelerated piston surface propagate along the positive C^+ characteristics and reach the shock front to be absorbed. Thus, Lick has considered that the positive characteristics are more important than the negative ones and so connected the shock front with the negative characteristic equation (19). Then equation (19) leads to the variation of H across the shock front. This approximation can be seen for a weak shock by comparing the above variation with one given due to equation (19). For the weak shock with the propagation velocity U and Mach number $M (= U/c_\phi)$, the shock conditions give

$$\frac{H - H_1}{c_{\phi 1}} \sim \frac{1-\nu^2}{16} \varepsilon^3 + O(\varepsilon^4), \quad (22)$$

$$\frac{\phi - \phi_1}{C_v} \sim \frac{\nu^2(1+\nu^2)}{3} \varepsilon^3 + O'(\varepsilon^4), \quad (23)$$

where $\varepsilon = M^2 - 1 \ll 1$ and subscript "2" is dropped out. From equation (19), (22) and (23)

it follows that

$$\left(\frac{dH}{d\phi}\right)_{c^-} / \left(\frac{dH}{d\phi}\right)_{w. \text{ Shock}} = \frac{4}{3}(1-\nu^2) \quad (24)$$

for the leading term in the Taylor series. (There is a misprint in Lick's eq. (2.14); the right-hand side should read $4(1-\nu^2)/3$.) For $\gamma=5/3$ ($\nu=0.5$), this is just unity. Furthermore, for even the other values of γ , this is nearly equal to unity. On the other hand, for a strong shock where $\varepsilon \gg 1$,

$$\left(\frac{dH}{d\phi}\right)_{c^-} / \left(\frac{dH}{d\phi}\right)_{s. \text{ Shock}} = \frac{1-\nu^2}{2} \left(1 + \frac{\nu}{\sqrt{1+\nu^2}}\right) \ln \varepsilon \quad (25)$$

for the leading term. This is proportional to $\ln \varepsilon$. Then we find that this approximation is wrong as ε is large.

As seen above, for the weak shock, it is concluded that the negative C^- characteristic equation (19) can be applied to the flow quantities behind the shock front. Then an expression is introduced as follows;

$$\sigma^- \equiv \frac{1}{U} \frac{\partial P_g}{\partial t} + \frac{\partial P_g}{\partial x} - \rho c_\phi \left(\frac{1}{U} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \right) \sim 0. \quad (26)$$

The negative characteristic equation (19) is re-written as

$$0 = \frac{1}{u-c_\phi} \frac{\partial P_g}{\partial t} + \frac{\partial P_g}{\partial x} - \rho c_\phi \left(\frac{1}{u-c_\phi} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \right). \quad (19)'$$

Subtracting equation (19)', we obtain the following expression;

$$\sigma^- = \left(\frac{1}{U} - \frac{1}{u-c_\phi} \right) \left(\frac{\partial P_g}{\partial t} - \rho c_\phi \frac{\partial u}{\partial t} \right). \quad (27)$$

Whence $|1/U - 1/(u-c_\phi)| \gg 1$, so $\partial P_g/\partial t - \rho c_\phi \partial u/\partial t$ must become nearly zero. Then we find that Lick's treatment is justified by the above condition.

2-4. The Method of Similarity Solution

This method is useful to get exact solutions for typical cases only where partial differential equations are transformed into ordinary ones depending on one argument (Sedov 1959; Zel'dovich and Raizer 1967). All physical quantities depend merely on the argument $\xi = rt^{-\beta}$, β being a rational number. Then both dimensions $[L]$ and $[T]$ cannot be separable. The ordinary ones are usually solved by numerical calculations. The exact solutions for the propagation of strong shocks are compared with those of the various methods mentioned above.

3. Improvement on the Characteristic Method

Saitô, M. and Saito, Y. (1977) have calculated the propagation of the shock wave formed by a very strong explosion and applied the solutions to an expanding shell near the galactic center. In this section, our method is re-examined in a slightly different

manner to provide facilities for comparison with the other methods.

The time scale of the explosion is shorter than that of the shock propagation. Thus the shock wave can be regarded as a shock pulse. The shock pulse propagates under the action of the gravitational field of the other mass, the differential rotation of gas itself and the circular magnetic field. Then the equation of continuity is re-written as

$$\frac{D\rho}{Dt} + \frac{\rho}{r} \frac{\partial}{\partial r}(ru) = 0, \quad (9)'$$

because $S(r)$ is proportional to r . The radial and azimuthal components of the equations of motion are

$$\frac{Du}{Dt} = -\frac{1}{r} \frac{\partial P}{\partial r} + \frac{v^2 - c_A^2}{r} - g(r), \quad (28)$$

$$\frac{Dv}{Dt} + \frac{uv}{r} = 0, \quad (29)$$

respectively. The induction equation of the azimuthal magnetic field is

$$\frac{DP_m}{Dt} + \rho c_A^2 \frac{\partial u}{\partial r} = 0. \quad (30)$$

Instead of the entropy equation, the polytropic relation is assumed as

$$\frac{D}{Dt}(P_g \rho^{-\gamma'}) = 0. \quad (31)$$

Here (r, θ) is the polar coordinate of which the axis and azimuthal direction are selected as usual, v the azimuthal velocity, $g(r)$ the specific gravitational force, P the total pressure, i. e., P_g plus the magnetic pressure $P_m \equiv B^2/(8\pi)$, c_A the Alfvén speed defined by $B/(4\pi\rho)^{1/2}$, and $c \equiv (\gamma' P_g/\rho)^{1/2}$ corresponds to the speed of sound. The value of γ' becomes γ for the adiabatic motion and becomes unity for the isothermal one. From equations (9)', (28), (30) and (31), we have obtained the characteristic equations,

$$D^+ P + \rho c_* D^+ u + \rho c_* \left\{ g(r) + \frac{1}{r} \left(c_A^2 - v^2 + \frac{c^2 u}{c_*} \right) \right\} = 0, \quad (32)$$

$$D^- P - \rho c_* D^- u - \rho c_* \left\{ g(r) + \frac{1}{r} \left(c_A^2 - v^2 - \frac{c^2 u}{c_*} \right) \right\} = 0, \quad (33)$$

where

$$D^+ = \frac{\partial}{\partial t} + (u + c_*) \frac{\partial}{\partial r}, \quad D^- = \frac{\partial}{\partial t} + (u - c_*) \frac{\partial}{\partial r}, \quad (34)$$

and $c_* = (c^2 + c_A^2)^{1/2}$ is the magnetosonic speed. From equation (9)' and (29)-(31), three conservation equations along the particale path line C^0 are derived;

$$rv = \text{const.}, \quad B/\rho r = \text{const.}, \quad P_g \rho^{-\gamma'} = \text{const.} \quad (35)$$

Now, the jump relations across the shock front,

$$v_2 = v_1 \text{ and } \frac{B_2}{\rho_2} = \frac{B_1}{\rho_1} \quad (36)$$

are added to the preceding Rankine-Hugoniot relations (2).

Since the motion of the piston which has caused the shock wave has already ceased, the motion of the gas is scarcely influenced by disturbances along the positive C^+ characteristics. Therefore, the positive characteristics are regarded as the minor ones and so non-dimensional quantity, χ is introduced, defined by the equation

$$\chi = \frac{r}{\rho_1(R)u^2} \left(\frac{\partial P}{\partial R} + \rho c_* \frac{\partial u}{\partial R} \right), \quad (37)$$

where R denotes the Lagrangian coordinate. It is remarked that χ is similar to the Riemann invariant J^+ in the expression of the Lagrangian coordinate R . This quantity indicates a magnitude of damping (for $\chi > 0$) or growing (for $\chi < 0$) of the shock wave mainly due to the motion of the piston. Then we have determined a sign of χ in the expanding gas behind the shock front. Since the gas is decelerated by the gravitational and magnetic forces as well as the pressure gradient, the values of $\partial P/\partial R$ and $\partial u/\partial R$ in equation (37) must be positive, i. e., $\chi > 0$.

Accordingly, Whitham's rule cannot be applied to the following reason; for a strong shock wave without the magnetic and gravitational forces, and the azimuthal velocity, σ^+ is expressed from (16), (32) and (37),

$$\sigma^+ = -\rho u \frac{u+c-U}{U} \left\{ \chi \frac{u}{R} + \frac{c^2}{r(u+c)} \right\}, \quad (38)$$

Whence $\chi u/R + c^2/\{r(u+c)\}$ is not small, so σ^+ is not infinitesimal. Therefore, the propagation of the shock pulse must be derived by the other method (Brinkley-Kirkwood 1947). Then we obtain an ordinary equation expressing the variation of the shock strength, P_2 ;

$$\begin{aligned} \frac{d \ln P_2}{d \ln R} &= \left(1 - \frac{P_1}{P_2} \right) \\ &\times \frac{-\chi \frac{u_2}{U} \left(1 - \frac{\rho_1 U}{\rho_2 c_*^2} \right) + A \frac{c_{A2}^2 - c_{A1}^2}{u_1 U} - \frac{c_1^2}{c_{*1} U}}{\frac{\rho_1 U}{\rho_2 c_*^2} + \frac{\partial \ln U}{\partial \ln P_1} + 1} \quad \text{on } \frac{dR}{dt} = U, \end{aligned} \quad (39)$$

with

$$\begin{aligned} A &= \frac{d \ln \rho_1}{d \ln R} + \frac{P_2}{P_2 - P_1} \left(1 + \frac{\partial \ln U}{\partial \ln P_1} + \frac{P_1}{P_2} \right) \frac{d \ln P_1}{d \ln R} \\ &+ \frac{\partial \ln U}{\partial \ln c_1} \cdot \frac{d \ln c_1}{d \ln R} + \frac{\partial \ln U}{\partial \ln \beta_{m1}} \cdot \frac{d \ln \beta_{m1}}{d \ln R}, \end{aligned} \quad (40)$$

$$\frac{\partial \ln U}{\partial \ln P_1} = -\frac{1}{2} \frac{P_2}{P_2 - P_1} + \frac{1}{2} \frac{P_2}{P_1} \left(\frac{\rho_1}{\rho_2} - \nu^2 \right) \left[\left(1 - \frac{\rho_1}{\rho_2} \right) \left(1 + \nu^2 \right) + (1 - 3\nu^2) \beta_{m1} \left(\frac{\rho_2}{\rho_1} \right)^2 + \frac{P_2}{P_1} - 1 \right], \quad (41)$$

$$\frac{\partial \ln U}{\partial \ln c_1} = 1, \quad (42)$$

$$\frac{\partial \ln U}{\partial \ln \beta_{m1}} = \frac{1}{2} \cdot \frac{\beta_{m1}}{1 - \beta_{m1}} + \frac{1}{2} (1 - 3\nu^2) \beta_{m1} \frac{\rho_2}{\rho_1} \left\{ 1 + \nu^2 + (1 - 3\nu^2) \beta_{m1} \left(\frac{\rho_2}{\rho_1} \right)^2 + \frac{P_2}{P_1} - 1 \right\}^{-1}, \quad (43)$$

where β_{m1} denotes the ratio of the magnetic to total pressures in the undisturbed gas.

Substituting equation (37) in the positive C^+ characteristic equation (32), we obtain

$$\frac{dP}{dt} + \rho c_* \frac{du}{dt} + \rho c_* \left\{ x \frac{u^2}{R} + g(r) - \frac{1}{r} \left(v^2 - c_A^2 - \frac{c^2 u}{c_*} \right) \right\} = 0 \text{ on } C^0, \quad (44)$$

with the relation of the force field in the undisturbed medium

$$g(r) = \frac{1}{r} \left(v_1^2 - c_{A1}^2 - \frac{P_1}{\rho_1} \frac{d \ln P_1}{d \ln R} \right). \quad (45)$$

The equation becomes a characteristic equation along the particle path line C^0 . This means that the positive (minor) characteristic equation is converted to the major characteristic one along the particle path line. The negative C^- characteristic equation (33) is re-written as follows;

$$\frac{dP}{dt} - \rho c_* \frac{du}{dt} - \rho c_* \left\{ g(r) - \frac{1}{r} \left(v^2 - c_A^2 + \frac{c^2 u}{c_*} \right) \right\} = 0 \text{ on } C^-. \quad (33)'$$

The entire flow field is determined from equations (33)', (39) and (44) by treating χ as a parameter.

The procedure for the determination of the flow field is as follows. A shock strength (P_2/P_1) is given at a starting point $X(r_1, t_1)$ in the (r, t) plane (see figure 4.) as an initial value together with a state of the undisturbed medium. For this shock strength and a tentatively given value of the parameter χ , equation (39) with the subsidiary relations (40)–(43) and the Rankine-Hugoniot relations (2), (35) uniquely specify the values of the physical quantities at a point $Y(r_2, t_2)$ just behind the shock front, where $t_2 = t_1 + \Delta t$. On the other hand, for this value of χ equations (33)' and (44) with equation (35) determine the values of the physical quantities at a point $Z(r_3, t_2)$ slightly inward of the front, provided that the boundary values on the C^- characteristic are found by interpolating the values at the points X and Y . The relations between the points X , Y , Z and the characteristic curves are illustrated in figure 4. Unless the resulting values of P , u , ρ and c_* at the points Y and Z satisfy equation (37), we give another value of χ and find again the corresponding values of the physical quantities at the same points. We continue this procedure for various values of χ until we find a self-consistent solution. After such a solution is found for point $X(r_1, t_1)$, the calculations mentioned above are

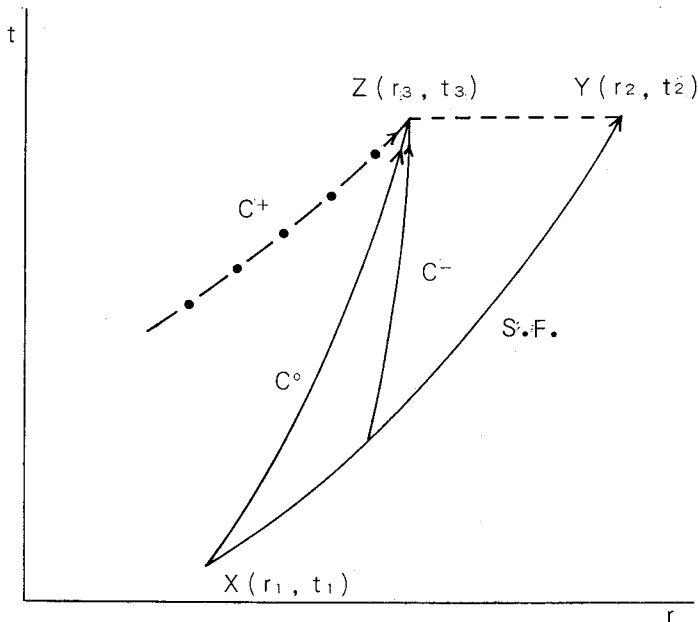


Fig. 4. Four characteristic direction in the (r, t) plane. S.F. denotes the path line of the shock front.

carried out the point $Y(r_2, t_2)$. Thus we can calculate the propagation of the shock-pulse step by step, determining the locally valid value of χ and finding simultaneously the gas motions slightly inward of the shock front.

Finally, we consider the structure of the shock-pulse. If a local value of χ and physical quantities slightly inward of the shock front found, the next value and quantities are determined from the above value and quantities by the same procedure. Repeating this procedure, we can determine physical states more inward of the front step by step. We get back a position where disturbances along the positive characteristics are valid compared with disturbances along the negative characteristics at the shock front.

4. Discussion and Conclusion

Although rather repeated, our method has been re-examined in a slightly different manner and the other methods of the shock propagation outlined briefly. Then it is valuable that our method is compared with the various methods on the basis of the facts mentioned in section 2 and the accuracy of numerical solutions explained by noting the effects of χ .

Substituting the Rankine-Hugoniot relations (2) which the subscript "2" is dropped out the left-hand side of equation (19), we see that the right-hand side is not generally zero. For only a weak shock wave, the right-hand side becomes approximately zero and the

value of χ nearly zero. For a strong shock wave, the flow field should be determined by using equations (11)', (18) and (37). Then the value of χ is fixed to be zero similar to the case for the weak shock. Therefore, the shock-expansion method (Lick 1966) reduces to Whitham's (1958) rule just behind the shock front. Similarly, if $\chi=0$, equation (39) reduces to the expression $\sigma^+=0$ due to Whitham's rule. This means that the rule is valid, provided that the relation,

$$\rho c \frac{\partial u}{\partial R} = -\frac{\partial P}{\partial R}, \quad (46)$$

is approximately satisfied behind the shock front. Sakurai (1960) has pointed out that Whitham's solutions are the first approximation of the similarity solutions for the strong shock wave which propagates through a plane non-magnetic medium with $\rho_0 \propto R^{-m}$, where m is constant. It is worth while noting that $\rho c \partial u / \partial R$ is nearly equal to $-(\partial P_g / \partial R)$ just behind the shock front in the solutions. On the other hand, Ôno, Sakashita and Ohyama (1961) have derived the propagation equation of a shock wave through a plane medium for a case with the magnetic force on the basis of Chisnell's method, in which the velocity of the shock front varies only by the effect of a stratification in undisturbed medium. Their treatment also implies the case $\chi=0$. For a strong shock wave with $\beta_{m1} \geq 0.1$ (the ratio of the magnetic to total pressures in the unperturbed medium), their results reduce to only one equation ;

$$d \ln P_2 = \left\{ 1 - \left(2 \frac{\rho_1}{\rho_2} \right)^{\frac{1}{2}} \right\} d \ln \rho_1 \quad \text{or} \quad (47)$$

$$d \ln U = - \left(\frac{1}{2} \frac{\rho_1}{\rho_2} \right)^{\frac{1}{2}} d \ln \rho_1.$$

This differential equation agrees with equation (39) for $\chi=0$ under the same conditions.

Let us determine the propagation velocity U of the strong shock. In the case of a plane homogeneous medium without gravitational and magnetic fields, the right-hand side in equation (39) has only the term including χ . Then equation (39) reduces to

$$d \ln U = -\frac{n}{2} d \ln r, \quad (48)$$

where n is a function of χ and ν ;

$$\frac{n}{2} = \frac{2\chi}{1-\nu^2} \left(1 - \frac{\nu}{\sqrt{1+\nu^2}} \right) \left(\frac{1}{2} + \frac{\nu}{\sqrt{1+\nu^2}} \right)^{-1} \quad (49)$$

Integrating equation (48), we obtain

$$U \propto r^{-\frac{n}{2}}. \quad (50)$$

Results of numerical calculation due to the techniques in section 3 show that the value of χ is 2.3 for $\gamma'=\gamma=5/3$ ($\nu=0.5$), i. e., $n=1$, and approaches to 1 as $\gamma'=\gamma \rightarrow 1$ ($\nu \rightarrow 0$), i.

e., $n=2$, and is 2.0 for $\gamma'=\gamma=7/5$ ($\nu=0.488$), i. e., $n=1.18$. It is noticed that the values of χ do not vary with distance, as long as the shock wave is very strong. Thus, as χ is not zero, the work is done behind the shock front, where the specific enthalpy of an element of the gas is increased.

These results agree with the limiting case in the similarity solutions given by Zel'dovich and Raizer (1967), in which $n=1$ for the energy conservation and $n=2$ for the momentum conservation. For $n=1.18$ ($\gamma'=\gamma=7/5$), let us consider the strong shock propagating through a homogeneous medium without the gravitational and magnetic forces. Then the propagation velocity U which is one of the physical quantities is integrated in an explicit form. The result is compared with them obtained by the method of similarity solution (Sedov 1959; Gross 1971), Chisnell's method, and Whitham's rule. The analytic solutions of U due to the first method based on the assumption of the energy conservation of the shock wave and rest of gas at origin are exact and are proportional to $r^{-3/2}$, $r^{-1/2}$ and $r^{1/2}$ for the spherical, cylindrical and plane parallel shocks, respectively. The propagation velocity due to our method for the cylindrical case is proportional to $r^{-0.59}$ and so is in good agreement with the corresponding similarity solution. Although the values of χ has not yet been determined for the other case, they agree probably with the corresponding ones. On the other hand, the variation laws of the propagation velocities due to both Chisnell's method and Whitham's rule are in the forms of $r^{-0.894}$, $r^{-0.197}$ and a constant for the above three kinds of shocks. Therefore, they do not agree with the corresponding solutions. As mentioned above, it can be concluded that our method for the motion of the shock-pulse improves on the other methods and rule.

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