Green's operators on multi-almost product manifolds

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Iutroduction. In the present paper, we shall study about Green's operators on a multi-almost product manifold. A multi-almost product structure is one of generalizations of an almost product structure defined in [1,3], and it seems to be closely related to a multifoliate structure defined by K. Kodaira and D. C. Spencer [2].

B. L. Reinhart [3] studied harmonic integrals on an almost product manifold. In [3], operators d', δ' and Δ' are given with respect to x alone and a Green's operator G' is defined and an analogue to Hodge's theorem is proved, i. e., it is proved that the d' cohomology is isomorphic to the space of Δ' harmonic forms. But the Green's operator G' is not completely continuous and the kernel of Δ' is infinite dimensional.

In § 1, we shall state the definition of multi-almost product structure, which is defined by giving m fields of subspaces of tangent spaces of the given manifold. And operators d_α $(\alpha=1,2,\cdots,m)$ and $\Delta_\alpha= d_\alpha\delta_\alpha+\delta_\alpha d_\alpha$ (δ_α is the adjoint operator of d_α). In § 2, we shall construct Green's operators for Δ_K.

§ 1 Multi-almost product structure

Let M be a connected paracompact (real) n-dimensional differentiahle manifold of class C^∞. Denote its tangent bundle by $T(M)$, the tangent P-vectors by $\Lambda^p T(M)$ $(p=1,2,\cdots,n)$ or $T^p(M)$, and a bundle of Grassmann algebras by $\Lambda T(M)=\sum_{p=0}^{n} T^p(M)$, where $\Lambda^0 T(M)$ is trivial bundle of real numbers, and $\Lambda^1 T(M)=T(M)$. Let $\Phi(M)$ be the bundle of cotangents, with other notations, $\Lambda^p\Phi(M)$, $\Lambda\Phi(M)$, etc., analogous to those for $T(M)$. Let d denote the operation of exterior differentiation on $\Lambda\Phi(M)$.

A *multi-almost product structure* is defined on M by giving m fields $S^1, S^2,\cdots, S^m$ (we assume that the integer m is in $0<m\leqq n$), ot class C^∞, of complementary proper subspaces of T_x. If we set $n_\alpha=\dim S^\alpha$ $(\alpha=1, 2,\cdots, m)$, then we have $n=\sum_\alpha n_\alpha$, where we assume $n_\alpha\neq 0$ for all α. Let P^α $(\alpha=1, 2,\cdots,m)$ be the projection of T_x onto S^α_x at every point $x\in M$, where S^α_x denotes the value of S^α at x.

In particular, if $m=2$, the multi-almost product structure on M reduces to an almost produet structure in the sense of [1,3]. A manifold with multi-almost product structure is said to be a *multi-almost product manifold.* Let M be a multi-almost product manifold. A multi-almost product sturcture on M induces the projection operators

$\Pi^*_{s_1,s_2,\dots,s_m}$ (resp. $\Pi_{s_1,s_2,\dots,s_m}$) in $\Lambda T(M)$ (resp. $\Lambda\Phi$). A vector field which is a cross-section of $\Pi^*_{s_1,s_2,\dots,s_m}\Lambda T(M) = T^{s_1 s_2\dots s_m}(M)$ is said to be of *type* $(s_1, s_2,\cdots,s_m)$ and similarly for $\Lambda\Phi(M)$. A map f of M into itself is said to be of *type* $(t_1, t_2,\cdots,t_m)$ with respect to the strncture if

$$\Pi^*_{s_1+s_1,\dots,s_m+t_m}\circ f^* = f^*\circ \Pi^*_{s_1,\dots s_m},$$

where $f^* : \Lambda T(M) \to \Lambda T(M)$ is the map induced by f, an operator L on $\Lambda T(M)$ or $\Lambda\Phi(M)$ may be decomposed into various types according to the definion:

$$\Pi_{t_1,t_2,\dots,t_m}L = \sum\nolimits_{s_1,s_2,\dots s_m} \Pi_{s_1+t_1,s_2+t_2,\dots,s_m+t_m}\circ L\circ \Pi_{s_1,s_2,\dots,s_m}.$$

Since the operator d of exterior differentiation is decomposed, so we find that

$$d = \sum_{\alpha,\beta=1}^{m} d_{\alpha\beta},$$

where $d_{\alpha\beta}$ is of type $(0,\cdots,0,\underset{\alpha}{-1},0,\cdots,0,\underset{\beta}{2},0,\cdots,0)$ for $\alpha<\beta$, of type $(0,\cdots,0,\underset{\alpha}{2},0,\cdots,0,\underset{\beta}{-1},0,\cdots,0)$ for $\alpha>\beta$, and of type $(0,\cdots,0,\underset{\alpha}{1},0,\cdots,0)$ for $\alpha=\beta$ respectively.

According to Guggenheim and Spencer [1], we may define an operator d_α : $\Phi^p(M) \to \Phi^{p+1}(M)$ by the axioms:

(1) If $\phi\in\Lambda^0\Phi_x(M)$, $v\in T_x(M)$ $(x\in M)$, then

$$(d_\alpha\phi)\ (v)=(P_\alpha v)\ (\phi).$$

(2) If $\phi\in\Lambda^0\Phi_x(M)$, then

$$(d_\alpha d+dd_\alpha)\,\phi=0.$$

(3) If $\phi\in\Lambda^p\Phi_x(M)$ anb $\psi\in\Lambda\Phi_x(M)$, then

$$(d_\alpha\ (\phi\wedge\psi)=d_\alpha\phi\wedge\psi+(-1)^p\phi\wedge d_\alpha\psi.$$

It is easily verified that $2\sum_{\beta=1}^{m} d_{\beta\alpha}-\sum_{\beta=1}^{m} d_{\alpha\beta}$, for fixed α, satisfieds these axioms hence we may set

$$d_\alpha = 2\sum_{\beta=1}^{m} d_{\beta\alpha} - \sum_{\beta=1}^{m} d_{\alpha\beta} \text{ for fixed } \alpha,$$

and find that $d=\sum_{\alpha=1}^{m} d_\alpha$.

In general, the operator d_α is not differential, i.e., $d_\alpha^2\neq 0$. It is, however, differential if and only if the multi-almost product structure is integrable, by which we mean that the Poisson bracket of two vector fields of type P_α is again a vcctor field of the same type. In this case, the Poincaré lemma is true for d_α and forms of type $(s_1, s_2,\cdots,s_\alpha,\cdots,s_m)$ with $s_\alpha>0$.

Let I_α be the increasingly ordered n_α-tuple $(i_1, i_2,\cdots,i_{n_\alpha})$ $(\alpha=1,2,\cdots,m)$. If a multi-almost product structure on M is integrable, we can apply the Frobenius' theorem to obtain local coordintates $(x_1^{I_1},\cdots,x_\alpha^{I_\alpha},\cdots,x_m^{I_m})$ in the neighbourhood of any point such that $P_\alpha T(M)$ is spanned by $\{\partial/\partial x_\alpha^{I_\alpha}\}$. This local coordinates is said to be a *local multi-*

almost product coordinate system, In terms of such a coordinate system, a differential form ϕ of type $(s_1, s_2, \cdots s_\alpha, \cdots, s_m)$ may be written as

$$\phi = \phi_{I_1(s_1)\cdots I_\alpha(s_\alpha)\cdots I_m(s_m)}\, dx_1^{I_1(s_1)} \wedge \cdots \wedge dx^{I_\alpha(s_\alpha)\alpha} \wedge \cdots \wedge dx_m^{I_m(s_m)},$$

where $I_\alpha\,(S_\alpha)$ is the increasingly ordered s_α-tuple $(i_1, i_2, \cdots, i_{s_\alpha})$ of integers in I_α and $dx_\alpha^{i\,\alpha(s_\alpha)} = dx_\alpha^{i_1} \wedge dx_\alpha^{i_2} \wedge \cdots \wedge dx_\alpha^{i_{s_\alpha}}$.

We notice that an integrable multi-almost prodect structure is a special case of a (real) multifoliate structure given by Kodaira and Spencer [2]. In fact, if a multifoliate structure on M has a finite totally ordered set instead of a finite partially ordered set P which defines a P-multifoliate set of integers and then if a local coordinate system $(x_1^{I_1}, \cdots, x_\alpha^{I_\alpha}, \cdots, x_m^{I_m})$ has a peoperty $\partial x_\alpha^i / \partial x_\beta^j = 0$ for $\alpha \neq \beta$. The multifoliate structure on M in this case becomes an integrable multi-almost product structure.

We may introduce on a multi-almost prodct manifold M a Riemannian metric such that the subspaces $P_\alpha T(M)$ $(\alpha = 1, 2, \cdots, m)$ are mutually orthogonal. If a multi-almost product structure on M is integrable and if local multi-almost product coordinates are used, the metric will have the form

$$ds^2 = \sum_{i,j=1}^{n_1} g_{ij}^{(1)}\, dx_1^j\, dx_1^j + \sum_{i,j=1}^{n_2} g_{ij}^{(2)}\, dx_2^i\, dx_2^j + \cdots + \sum_{i,j=1}^{n_m} g_{ij}^{(m)}\, dx_m^i\, dx_m^j,$$

where components of metric teusor $g_{ij}^{(\alpha)}$ $(\alpha = 1, 2, \cdots, m)$ are C^∞ functions of $(x_1^{I_1}, x_2^{I_2}, \cdots, x_m^{I_m})$. This metric is said to be a *multi-almost prodvct metric.*

Hereafter we assume that the manifold M has an integrable multi-almost product strucfure and a multi-almost product metric.

We may define an operator δ_α for a form ϕ with totally degree q by

$$\delta_\alpha \phi = (-1)^{nq+n+1} * d_\alpha * \phi,$$

where $*$ is the duality operator deiined by the given metric. Since $\delta\phi = (-1)^{nq+n+1} * d * \phi$, we have $\delta = \sum_{\alpha=1}^{m} \delta_\alpha$. Let K be the increasingly ordered p tuple $(\lambda, \mu, \cdots, \nu)$ $(0 < p \leqq m;\ \lambda, \mu, \cdots, \nu \in \{1, 2, \cdots, \mathrm{m}\})$ and by $\bar{K}$ we denote the increasingly ordered $(m-p)$-tuple consisting of the subset of $\{1, 2, \cdots, m\}$ complementary to K. We may define special Laplacian operators by

$$\begin{aligned} \Delta_\alpha &= d_\alpha \delta_\alpha + \delta_\alpha d_\alpha \quad (\alpha = 1, 2, \cdots, m), \\ \tilde{\Delta}_K &= \Delta_\lambda + \Delta_\mu + \cdots + \Delta_\nu, \\ \tilde{\Delta} &= \Delta_1 + \Delta_2 + \cdots + \Delta_m. \end{aligned}$$

It is easily showed that Δ_α, $\tilde{\Delta}_K$ and $\tilde{\Delta}$ preserve types, i.e., they are of type $(0, 0, \cdots, 0)$. We denote a usual Laplacian operator by Δ.

If the manifold M is compact, the inner product (ϕ, ψ) is defined in usual way for $\phi, \psi \in \Phi(M)$ and the norm $\phi=\sqrt{(\phi,\phi)}$ is defined.

Let M be a compact multi-almost product manifold with integrable structure. A Riemannian metric on M is said to be *torsiouless* if $g_{ij}^{(\alpha)}$ $(\alpha=1,2,\cdots,m)$ are C^∞ functions of $x_\alpha^{I\alpha}$ $(\alpha=1,2,\cdots,m)$ alone. Setting

$$d_K=d_\lambda+d_\mu+\cdots+d_\nu,\quad \delta_K=\delta_\lambda+\delta_\mu+\cdots+\delta_\nu$$

and

$$\Delta_K=d_K\delta_K+\delta_K d_K,$$

we have the following proposition.

PROPOSITION 1. *If the metric on M is torsionless, then* $\tilde{\Delta}_K=\Delta_K$ *holds good and moreover* $\Delta=\tilde{\Delta}$.

Since the metric is torsionless we ean prove by the method similar to Reinhart [3].

PROPOSITION 2. *If M is an integrable multi-almost product manifold, we have* $d_K^2=0$ *and* $d_{\bar{K}}^2=0$.

In fact, $d^2=0$ hence $d_\lambda d_\mu=0$ for all $\lambda, \mu \in \{1, 2,\cdots,m\}$. On the other hand, $d_\alpha^2=0$ for all $\alpha\in\{1,2,\cdots,m\}$ since the structure is integrable. Then the proposition is proved.

Then we have

$$\Delta = \tilde{\Delta} = \Delta_K + \Delta_{\bar{K}}$$

for a torsionless integrable multi-almost product manifold M.

§ 2 Green's operators for Δ_K

Let $\mathfrak{L}_K^r$ be a sheaf of germs of square integrable forms of degree r which depend on x_K alone, where $x_K=\{x_\lambda^{I\lambda}, x_\mu^{I\mu},\cdots,x_\nu^{I\nu}\}$, and $\mathfrak{L}_{\bar{K}}^s$ is defined similarly. Let $\mathfrak{L}_{K\bar{K}}$ be the direct sum $\sum_{r,s} \mathfrak{L}_K^r \wedge \mathfrak{L}_{\bar{K}}^s$. By the method similar to Reinhart [3], we can construct a coherent subsheaf $\mathfrak{B}_K$ of $\mathfrak{L}_{K\bar{K}}$ such that $\phi_1, \phi_2,\cdots,\phi_t$ are sections of $\mathfrak{B}_K$, where $\phi_1, \phi_2,\cdots,\phi_t$ are also sections of $\mathfrak{L}_{K\bar{K}}$. And any ϕ is a stalk of $\mathfrak{B}_K$ is expressed as $\sum \phi_i \wedge \gamma_i$ in some neighbourhood, where γ_i is one of finite set of sections of $\mathfrak{L}_{\bar{K}}^s$.

Next we shall define the second inner product of forms on M. It is defined by

$$((\phi,\varphi))_U=(\phi, \varphi)+\sum_{\alpha,j}\int_{U_j}\sum_i \frac{\partial\phi_{IK(SK)I\bar{K}(S\bar{K})}}{\partial x_\alpha^i}\frac{\partial\varphi_{IK(SK)I\bar{K}(S\bar{K})}}{\partial x_\alpha^i}\,dx_K\,dx_{\bar{K}}$$

and the corresponding norm by $\|\phi\|_U=\sqrt{((\phi, \phi))_U}$, where (ϕ, ψ) is the usual inner product and $\mathfrak{U}$ a finite covering $\{U_i\}$ of M.

The space $\mathbf{L}_2(\mathfrak{B}_K)$ is the completion in the usual norm of the set of C^∞ sections of $\mathfrak{B}_K$; the space $\mathbf{P}_2(\mathfrak{B}_K)$ is the completion in the second norm. Let $H_\alpha(\mathfrak{B}_K)$ $(\alpha \in K)$ be a subspace consisting of forms ω in $\mathbf{P}_2(\mathfrak{B}_K)$ satisfying $d_\alpha\omega = \delta_\alpha\omega = 0$ and $H_K(\mathfrak{B}_K) = H_\lambda(\mathfrak{B}_K) \cap H_\mu(\mathfrak{B}_K) \cap \cdots \cap H_\nu(\mathfrak{B}_K)$. We have the following theorems by the method similar to [3] and the proof is omitted.

THEOREM 1. *Let M be a compact multi-almost product manifold with torsionless metric and let $\mathfrak{B}_K$ be a coherent svbsheaf of $\mathfrak{L}_{K\bar{K}}$. Let ω_0 be a form in $\mathbf{L}_2(\mathfrak{B}_K)$ orthogonal to $H_K(\mathfrak{B}_K)$ such that*

$$\sum_{\alpha\in K} \{(d_\alpha\Omega_0, d_\alpha\zeta) + (\delta_\alpha\Omega_0, \delta_\alpha\zeta)\} = (\omega_0, \zeta)$$

for all $\zeta \in \mathbf{P}_2(\mathfrak{B}_K)$. Furthermore, the from ω_0 to Ω_0 is a bounded linear transformation from $\mathbf{L}_2(\mathfrak{B}_K)$ into $\mathbf{P}_2(\mathfrak{B}_K)$.

THEOREM 2. *Let M be a compact multi-almost product manifold with torsionless metric, aud $\mathfrak{B}_K$ be a coherent subsheaf of $\mathfrak{L}_{K\bar{K}}$. If $(\Omega_0, \Delta_K\zeta) = (\omega_0, \zeta)$ for all ζ with sufficiently small support in $\mathbf{L}_2(\mathfrak{B}_K)$ which are C^∞, and if ω_0 is in $\mathbf{L}_2(\mathfrak{B}_K)$ and C^∞, then Ω_0 is C^∞.*

Combining Theorem 1 and 2, we can construct the Green's operator for Δ_K. By the method of M. H. Stone we can show that $\Delta_\alpha = d_\alpha\delta_\alpha + \delta_\alpha d_\alpha$ is a self-adjoint operator, in particular is also closed, hence so is Δ_K.

THEOREM 3. *There is defiucd on $\mathbf{L}_2(\mathfrak{B}_K)$ a bounded symmetric operator $G_{K},\mathfrak{B}_K$ such that $\Delta_K G_K\, \mathfrak{B}_K\phi = G_K,\mathfrak{B}_K\Delta_K\phi = \phi - H_K,\mathfrak{B}_K\phi$ and $G_K, \mathfrak{B}_K\, H_K,\mathfrak{B}_K\phi = H_K,\mathfrak{B}_K\, G_K,\mathfrak{B}_K\phi = 0$, where $H_K,\mathfrak{B}_K$ is the projection to the space $H_K(\mathfrak{B}_K)$.*

PROOF, Let ω_0 be orthogonal to $H_K(\mathfrak{B}_K) = H_\lambda(\mathfrak{B}_K) \cap H_\mu(\mathfrak{B}_K) \cap \cdots \cap H_\nu(\mathfrak{B}_K)$. By Theorem 1, We can find Ω_0 orthogonal to $H_K(\mathfrak{B}_K)$ such that $\sum (d_\alpha\Omega_0, d_\alpha\zeta) + \sum (\delta_\alpha\Omega_0, \delta_\alpha\zeta) = (\omega_0, \zeta)$ for all $\zeta \in \mathbf{P}_2(\mathfrak{B}_K)$. Moreover if ω_0 is C^∞ by Theorem 2, Ω_0 is C^∞, whence $\Delta_K\Omega_0 = \omega_0$. Define $G_K\mathfrak{B}_K(\omega_0) = \Omega_0$, then $\Delta_K G_K,\mathfrak{B}_K\omega_0 = \omega_0$. Let $H_K,\mathfrak{B}_K$ be a projection on $H_K(\mathfrak{B}_K)$. If $H_K,\mathfrak{B}_K\phi$ is C^∞ for any ϕ, so that $\phi - H_K,\mathfrak{B}_K$ is C^∞ if ϕ is. Let ϕ be an arbitrary form in $\mathbf{L}_2(\mathfrak{B}_K)$, and let $\{\phi_{\alpha i}\}$ be a sequence of C^∞ forms approximating ϕ. Then $\{\omega_i\} = \{\phi_i - H_K,\mathfrak{B}_K\phi_i\}$ is a sequence of C^∞-forms approximating $\omega = \phi - H_K,\mathfrak{B}_K\phi$ and each ω_i satisfies $\Delta_K G_K,\mathfrak{B}_K\omega_i = \omega_i$. Since $|G_K,\mathfrak{B}_K\omega_0| \leqq \| G_K,\mathfrak{B}_K\omega_0 \| \leqq C|\omega_0|$, $G_K,\mathfrak{B}_K$ is bounded. Hence $G_K,\mathfrak{B}_K\omega_i \to G_K,\mathfrak{B}_K\omega$, while $\Delta_K G_K,\mathfrak{B}_K\omega_i = \omega_i \to \omega$. Since Δ_K is closed, we have $\Delta_K G_K,\mathfrak{B}_K\omega = \omega$. Now let us define $G_K,\mathfrak{B}_K\phi = G_K,\mathfrak{B}_K\omega$. Then $\Delta_K,\mathfrak{B}_K\, G_K,\mathfrak{B}_K\phi = \phi - H_K,\mathfrak{B}_K\phi$. If $H_K(\mathfrak{B}_K)$, we have $G_K,\mathfrak{B}_K\phi = 0$, hence $G_K,\mathfrak{B}_K G_K,\mathfrak{B}_K\phi = 0$. Also $H_K,\mathfrak{B}_K G_K,\mathfrak{B}_K\phi = 0$. Hence $G_K,\mathfrak{B}_K$ is symmetric on $\mathbf{L}_2(\mathfrak{B}_K)$ and $G_K,\mathfrak{B}_K\, \Delta_K\phi = \phi - H_K,\mathfrak{B}_K\phi$ for all ϕ in the domain of Δ_K. q.e.d.

By a method similar to Reinhart, we can show that the Green's operator is independent from the choice of coherent subsheaves of $\mathfrak{L}_{KK}$.

THEOREM 4, *Let M be a compact multi-aimost product manifold with torsionless* *metric. Let ϕ be a C^∞ section of the sheaf $\mathfrak{L}_{K\bar{K}}$. Then there is a C^∞ section $G_K\phi$ of $\mathfrak{L}_{K\bar{K}}$ such that $\Delta_K G_K\phi = G_K\Delta_K\phi = \phi - H_K\phi$ and $G_K H_K\phi = H_K G_K\phi = 0$. In these equations, $G_K\phi = G_{K,\mathfrak{B}_K}\phi$ and $H_K\phi = H_{K,\mathfrak{B}_K}\phi$, where $\mathfrak{B}_K$ is dn arbitrary coherent subsheaf of $\mathfrak{L}_{K\bar{K}}$ having ϕ as a section.*

The operator G_K appeared in Theorem 4 is called a *Green's operator* for Δ_K. G_K is not in general a bounded operator on the Hilbert space of all square integrable forms on *M*, since G_K may be densely defined without being everywhere defined. We can easily show that $\Delta_K d_K = d_K \Delta_K$ and G_K commutes with d_K and δ_K. Hence we have an analogue to Hodge's theorem.

THEOREM 5. *Let M be a compact multi-almost product manifold with torsionless metric. Then the d_K cohomology of sections of the sheaf $\mathfrak{L}_{K\bar{K}}$ is isomorphic to the space of Δ_K harmonic sections of $\mathfrak{L}_{K\bar{K}}$, the isomorphism being given by assigning to each cohomology class the unique harmonic form contained in it.*

Next we shall consider a special case. We consider a Green's operator for $\tilde{\Delta}$. In this case, We can construct the Green's operator by a method similar to Reinhart [3], and we can prove the following result.

PROPOSITION 3. *Let M be a compact multi-almost product manifold. There exists a symmetric and completely continuous operator $\tilde{G}$ which satisfies*

$$\beta = \tilde{\Delta}\,\tilde{G}\,\beta + \tilde{H}\,\beta = \tilde{G}\,\tilde{\Delta}\,\beta + \tilde{H}\,\beta$$

and $\tilde{G}\tilde{H} = \tilde{H}\tilde{G} = 0$, where $\tilde{H}\beta$ is the projection of β on the space $\tilde{H} = \{\phi\ \tilde{\Delta}\phi = 0\}$ and β is a C^∞ form in $\mathfrak{L}_2(M)$.

The Green's operator $\tilde{G}$ for $\tilde{\Delta}$ fails to commute with *d*, d_α and related operators.

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