On infinitesimal automorphisms of locally product Riemannian spaces

TAKUYA SAEKI

§0. INTRODUCTION. A locally product Riemannian space is formally analogous to a Kählerian space in some sense. Let us consider an n-dimensional *locally product Riemansnian space*. Then, by definition, there exist a system of coordinate neighborhoods $\{U\}$ such that in each U the line element is given by the form

$$ds^2 = g_{ab}(x^c)\, dx^a dx^b + g_{ij}(x^k) dx^i dx^j \tag{0.1}$$

and in $U \cap U'$ the coordinte transformation $(x^a, x^i) \rightarrow (x^{a'}, x^{i'})$ is given by $x^{a'} = x^{a'}(x^b)$, $x^{i'} = x^{i'}(x^j)$, where the Greek indices $\lambda, \mu, \cdots$ run over the range $1, 2, \cdots, n$ and the Latin indices $a, b, \cdots$ (resp. $i, j \cdots$) run over the range $1, 2, \cdots, p$ resp. $p+1, p+2, \cdots, p+q=n$). Such a coordinate system (x^a, x^i) will be called a separating coordinate system.

If we define by

$$(\varphi_\mu^\lambda) = \begin{pmatrix} \delta_b^a & 0 \\ 0 & -\delta_j^i \end{pmatrix} \tag{0.2}$$

in each U, then they define a tensor field over a locally product Riemannian space and satisfy

$$\text{a)}\varphi_\mu^\nu \varphi_\nu^\lambda = \delta_\mu^\lambda, \ \text{b)} g_{\nu\lambda}\varphi_\mu^\nu = g_{\mu\nu}\varphi_\lambda^\nu, \ \text{c)}\nabla_\lambda \varphi_\nu^\mu = 0\,, \tag{0.3}$$

where ∇_λ denotes the operator of the Riemannian covariant derivative. (0,3) a) shows that φ_μ^λ assigns an *almost-product structure* to the space.

Under the above situation, Reinhart [9] has introduced operators d', δ' and $\triangle' = d'\delta' + \delta' d'$ on differential forms on the space, all of them differentiating with respect to x^a alone. Under some additional conditions, he has studied the operator $\triangle'$ and defined Green's operator G' and proved an analogous theorem of Hodge theorem which asserts that each cohomology class contains one and only one form satisfying $\triangle'\alpha = 0$.

On the other hand, in a Riemannian space some of the infinitesimal automorphisms, i.e., infinitesimal isometries and infinitesimal conformal transformations are characterized by the usual operators d, δ and $\triangle$ [2, 7]. Therefore in the present paper, we shall ask the relations between the special operators d', δ', $\triangle'$, δ'', δ'' and $\triangle''$ and the infinitesimal automorphisms.

§1. Preliminary remarks

1.1. Now and in the sequel, let M be a connected paracompact differentiable manifold of dimension n and of class C^∞. Let M be also a locally product Riemannian space, for brevity, an LPR-space. Let α be a (p, q)-form[1] on M represented locally by

$$\alpha = \alpha_{(a_1\cdots a_p i_1\cdots i_q)} dx^{a_1} \wedge \cdots \wedge dx^{a_p} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_q},$$

where now and in thee sequel $(a_1 \cdots a_p i_1 \cdots i_q)$ implies $a_1 < \cdots < a_p < i_1 < \cdots < i_q$. If we write

$$d'\alpha = (d'\alpha)_{(a_1\cdots a_{p+1} i_1\cdots i_q)} dx^{a_1} \wedge \cdots \wedge dx^{a_{p+1}} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_q} \tag{1.1}$$

then

$$(d'\alpha)_{a_1\cdots a_{p+1} i_1\cdots i_q} = \delta^{c(b_1\cdots b_p)}_{a_1\cdots a_{p+1}} \nabla_c \alpha_{(b_1\cdots b_p) i_1\cdots i_q} \tag{1.2}$$

and

$$\delta'\alpha = (\delta'\alpha)_{(a_1\cdots a_{p-1} i_1\cdots i_q)} dx^{a_1} \wedge \cdots \wedge dx^{a_{p-1}} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_q}, \tag{1.3}$$

where

$$(\delta'\alpha)_{a_1\cdots a_{p-1} i_1\cdots i_q} = -g^{bc} \delta^{(c_1\cdots c_p)}_{b a_1\cdots a_{p-1}} \nabla_c \alpha_{(c_1\cdots c_p) i_1\cdots i_q}. \tag{1.4}$$

Then, the special Laplacian operator

$$\triangle' = d'\delta' + \delta' d'$$

is given by

$$\begin{aligned}(\triangle'\alpha)_{a_1\cdots a_p i_1\cdots i_q} &= -g^{bc}\nabla_b\nabla_c \alpha_{a_1\cdots a_p i_1\cdots i_q} \\ &\quad + \sum_{\rho=1}^{p} \alpha_{a_1\cdots a_{\rho-1} b a_{\rho+1}\cdots a_p i_1\cdots i_q} R^{b}_{a_\rho} \\ &\quad + \frac{1}{2}\sum_{\sigma=1}^{p}\sum_{\rho=1}^{p} \alpha_{a_1\cdots a_{\rho-1} b a_{\rho+1}\cdots a_{\sigma-1} c a_{\sigma+1}\cdots a_p i_1\cdots i_q} R^{bc}{}_{a_\rho a_\sigma}.\end{aligned} \tag{1.5}$$

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1) See Reinhart [9].

Similarly we can define operatore d'', δ'' and $\triangle''$ with regard to variables (x^i).

1.2. According to Reinhart [9], we introduce the sheaves E and B on a compact LPR-space. Let D_1^r be the sheaf of germs of r-forms which depend on the first type variables, i.e., the variables (x^a) and similarly for D_2^s. Let E be the direct sum $\sum_{r,s} D_1^r \wedge D_2^s$. Thus there exists a coherent subsheaf B of E such that any section of B depends on a finite number of fixed forms in the second variables and that a finite set of sections of E is sections of B. It is known the following theorem [9].

THEOREM 1. (Reinhart) *Let M be a compact LPR-space. Then the space of d'-cohomology classes of sections of the sheaf E on M is isomrphic to the space of $\triangle'$-harmonic sections of E, i.e., $\triangle'\alpha = 0$, the isomorphism being given by assinging to each cohomology class the unique $\triangle'$-harmonic form contained in it.*

This theorem is an analogue to Hodge theorem. In the proof of the theorem, Reinhart defined the Green's operator G' for $\triangle'$ and showed the formula

$$\alpha = d'\delta'G'\alpha + \delta'd'G'\alpha + H'\alpha, \tag{1.6}$$

where $H'\alpha = H'_B\alpha$ is the projection on the kernel of $\triangle'$ for $\alpha \in \mathfrak{L}_2(B)$ and $\mathfrak{L}_2(B)$ is the space of the completion in a usual norm of the set of C^∞ sections of B.

§2. CURVATURE AND LPR-SPACES

2.1. Let us discuss the relations between the d′-cohomology and the curvature of M.

LEMMA. *Let ϕ be a $(1,0)$-form with a compact carrier on an orientable and integrable LPR-space. Then we obtain*

$$\int_M \delta'\phi dM = 0,$$

where dM denotes the volume element of M.

In fact, since ϕ is of type $(1,0)$, $\int_M \delta'\phi dM = \int_M \delta\phi dM = 0$ by the Green's theorem.

Let $R = (R_{\alpha\beta})$ be a Ricci curvature tensor with respect to a Riemannien connection on M. We employ the notation $< t, t' >$ to mean the local scalar product of the covariant tensors t and t'. We may introduce the Ricci operator S on 1-forms ϕ of M by

$$S : \phi_\alpha \longrightarrow 2R_\alpha^\beta \phi_\beta .$$

The quadratic form $< S\phi, \phi >$ is called a *Ricci quabratic form.*

Then we have $R_{\alpha\beta}\phi^\alpha\phi^\beta = R_{ab}\phi^a\phi^b + R_{ij}\phi^i\phi^j$. The quadratic form $R_{ab}\phi^a\phi^b$ (resp. $R_{ij}\phi^i\phi^j$) is called the *first* (resp. *second*) *Ricci quadratic form.*

Let ϕ be a (1,0)-form on M. The Riemannian metric on M admits to identify a contravariant vector field of type (1,0) with a (1,0)-form. We have

$$0 = (\triangle'\phi, \phi) = \int_M \left\{ -(\nabla^a\nabla_a\phi_b)\phi^b + R_{ab}\phi^a\phi^b \right\} dM$$

for a $\triangle'$-harmonic ϕ and a compact orientable M. On the other hand, by the above Lemma,

$$0 = \int_M \delta' d' < \phi, \phi > dM = -2\int_M \left\{ \phi^b\nabla^a\nabla_a\phi_b + \nabla^a\phi^b\nabla_a\phi_b \right\} dM,$$

hence we have

$$\int_M R_{ab}\phi^a\phi^b dM = -\int_M \nabla^b\phi^a\nabla_b\phi_a dM. \tag{2.1}$$

Assum that the first Ricci quadratic form is strictly positive definite, then the (1,0)-form ϕ vanishes. Hence the following conclusion.

PROPOSITION 1. *Let M be a compact and orientable LPR-space. Assume that the first Ricci quadratic form with respect to a Riemannian connection on M is everywhere positive definite. Then there exists no form ϕ of type* (1,0) *such that* $\triangle'\phi = 0$.

If ϕ is a C^∞ section of the sheaf E, we have an analogue to Myers-Bochner's theorem.

THEOREM 2. *Let M be a compact and orientable LPR-space. Assume that the first Ricci quadratic form with respect to a Riemannian connection on M is everywhere positive definite. Then the dimension of the space of the d'-cohomology classes consisting of sections of type* (1,0) *of the sheaf E is zero.*

Proof. Let ϕ be a C^∞ section of type (1,0) of the sheaf E and let B any coherent subsheaf of E having ϕ as a section of B considered in Paragraph 1.2. For ϕ, Proposition 1 holds . By Theorem 1 we can get the theorem. q.e.d.

Moreover the formula (2.1) shows that in a compact and orientable LPR-space M a $\triangle'$-harmonic vector field of type (1,0) for which the first Ricci quadratic form is positive semi-definite is necessarily a parallel vector field.

2.2. We now seek a result analogous to Theorem 2 for general case. Let ϕ be a (r, s)-form such that $\triangle'\phi = 0$. Since

$$\int_M d'\delta' < \phi, \phi > dM = 0,$$

we obtain the formula

$$\int_M \Big\{ rR_{ab}\phi^{aa_2\cdots a_r j_1\cdots j_s}\phi^{b}{}_{a_2\cdots a_r j_1\cdots j_s} + \frac{r(r-1)}{2} R_{abcb}\phi^{aba_3\cdots a_r j_1\cdots j_s}\phi^{cd}{}_{a_3\cdots a_r j_1\cdots a_s} + \nabla_a\phi_{a_1\cdots a_r j_1\cdots j_s}\nabla^a\phi^{a_1\cdots a_r j_1\cdots j_s}\Big\} dM = 0$$

by a method similar to [2]. Setting

$$F'(\phi) = R_{ab}\phi^{aa_2\cdots a_r j_1\cdots j_s}\phi^{b}{}_{a_2\cdots a_r j_1\cdots j_s} + \frac{r-1}{2} R_{abcd}\phi^{aba_3\cdots a_r j_1\cdots j_s}\phi^{cd}{}_{a_3\cdots a_r j_1\cdots j_s},$$

we have

$$r\int_M F'(\phi)dM = -\int \nabla_a\phi_{a_1\cdots a_r j_1\cdots j_s}\nabla^a\phi^{a_1\cdots a_r j_1\cdots j_s} dM < 0$$

since $(\triangle'\phi, \phi) = 0$. Then we obtain

THEOREM 3. *Let M be a compact and orientable LPR-spabe. Assume that the quadratic form $F'(\phi)$ is positive definite for all differential forms ϕ on M. Then there exists no $\triangle'$-harmonic forms on M.*

§3. INFINITESIMAL AUTOMORPHISMS

Let M be an LPR-space. A diffeomorphism of M which leaves a induced almost-product structure invariant is said to be an *almost-product transformation*. Since the Lie algebra $\mathfrak{gl}(p,q)$ of a Lie subgroup $GL(p,q)$ contains a matrix of rank one, by the theorem in [8, 10], the group of almost-product transformation of M is not a Lie group. But we can define an infinitesimal almost-product transformation. It is called that a vector field $X = X^\alpha(\partial/\partial x^\alpha)$ on M generates *infinitesimal almost-product transformations* or, for brevity, X is an *AP-transformation* if $(\partial^\beta X^\alpha) \in \mathfrak{gl}(p,q)$, where $(\partial/\partial x^\alpha)$ is the distinguished frame. A vector field X said to be *decomposable* if X generates AP-transformations.

Let $(X^a, X^i) \equiv (X^1, \cdots X^p, X^{p+1}, \cdots, X^n)$, $(X^a, 0)$ and $(0, X^i)$ be components of vector fields X, X' and X'' respectively with respect to a distinguished frame. It is called that a vector field X' generates *infinitesimal product isometries of the first type* or, for brevity, X' is a *P_1-isometry* if $\pounds(X')g = 0$, where $\pounds(X')$ denotes the Lie derivative with respect to X'. Similarly we can define a P_2-

isometry with respect to X''. It is called that a vector field $X(=X'+X'')$ generates *infinitesimal product isometries* or, for drevity, X is a *P-isometry* if $\mathfrak{L}(X')g=\mathfrak{L}(X'')=0$.

It is easily show that if X is a P-isometry X is decomposable. If X is a P-isometry, X is an infinitesimal isometry. Conversely if X is a decomposable vector field an infinitesimal isometry X is a P-isometry.

Let $\rho(X^a)$ be a C^∞ function such that $\partial\rho(X^a)=0$. It is called that X' generates *infinitesimal product conformal transformations of the first type* or, for brevity, X' is a *PC_1-transformation* if $\mathfrak{L}(X')g=\rho g$. Similarly we can define a *PC_2-transformation* with respect to X'' aad a C^∞ function $\sigma(x^i)$ such that $\partial_a\sigma(x^i)=0$. It is called that $X(=X'+X'')$ generates *infinitesimal product conformal transformations* or, for brevity, X is a *PC-transformation* if $\mathfrak{L}(X')g=\rho g$ and if $\mathfrak{L}(X'')g=\sigma g$.

If a vector field X is a PC-transformation, X is decomposable. Moreover if a vector field X is a PC-transformation and if $\rho=\sigma=$ constant, X is decomposable and is an *infinitesimal homothetic transformation.* It is clear that the Poisson bracket of two decomposable vector fields is also decomposable. Then all the infinitesimal automorphisms above defined admit a structure of Lie algebras.

§4. PROPERTIES OF PC-TRANSFORMATIONS

4.1. Let M be an LPR-space. The Riemannian metric on M admits to identity a rector field X with a 1-form ξ on M. Hence we can define the Lie derivative with respect to a vector field defined by and it is denoted by $\mathfrak{L}(\xi)$. If ξ is a 1-form on M, ξ is decomposed into two forms ξ' and ξ'', where ξ' is a form of type $(1,0)$ and ξ'' of type $(0,1)$.

We cnn easily show the following results by the method similar to [7] and the proofs are omitted.

According to [7], we shall define a symmetric tensor field $t(\xi')$ and $t(\xi'')$ by

$$t(\xi')_{ab}=\nabla_a\xi'_b+\nabla_b\xi'_a+\frac{2}{p}\delta'\xi'g'_{ab},$$

$$t(\xi')_{ib}=t(\xi')_{bi}=\nabla_i\xi'_b,$$

$$t(\xi')_{ij}=0,$$

and similarly for $t(\xi'')$.

PROPOSITION 1. *On an LPR-space, a $(1,0)$-form ξ' is a PC_1-transformation if and only if ξ' satisfies $t(\xi')=0$.*

For a (0,1)-form ξ'', We can get a similar proposition. By Propostion 1, we have the following characterization.

PROPOSITION 2. *On an LPR-space, in order that a 1-form $\xi(=\xi'+\xi'')$ is a PC-transformation, it is necessary and sufficient that ξ' satisfies $t(\xi')=0$*

and ξ'' *satisfies* $t(\xi'')=0$, *where* ξ' *is a* $(1,0)$*-form and* ξ'' *a* $(0,1)$*-form, provid* $p>1$ *and* $q>1$.

PROPOSITION 3. *Let* ξ *be a* 1*-form generating PC-transformations on an LPR-space M. In order that* ξ *is a P-isometry, it is necessary and sufficient that* ξ *satisfies* $\delta'\xi'=\delta''\xi''=0$, *where* ξ' *(resp.* ξ''*) is a* $(1,0)$*-form (resp.* $(0,1)$*-form) with* $\xi=\xi'+\xi''$.

4.2. In this paragraph, we assume that M is compact and orientable. Let L be a subalgebra of Lie algebra T of tangent vector fields on M. A p-form on M is said to be *L-invariant* if it is a zero of all the derivations $\mathfrak{L}(X)$ for $X\in L$. We shall consider $\triangle'$-harmonic forms on sections of E defined in §1. We can easily show the following propositions.

PROPOSITION 4. *The* $\triangle'$*-harmonic sections of the sheaf E defined in Paragraph* 1.3 *on a compact and orientable LPR-space M are L-invariant, where L is the Lie algebra of* P_1*-isometries on M.*

PROPOSITION 5. *On a compact and orientable LPR-space, a* $\triangle'$*-harmonic* (r,s)*-section* ϕ *of the sheaf E is L-invariant, if and only if,* $P=2r$ *or,* $\delta'\xi'\cdot\phi$ *is* δ'*-closed, where L is the Lie algebra of* PC_1*-transformations.*

4.3. In this paragraph we characterize the PC-transformations of a compact and orientable M as solutions of a system of differential equations on M.

Let $t(\xi')$ and $t(\xi'')$ be symmetric tensor fields defined in Paragraph 4.1. We shall calculate the covariant derivative of $t(\xi')$ and $t(\xi'')$. By the Ricci identity, we have

$$\nabla^a t(\xi')_{ab}=\nabla^a\nabla_a\xi'_b+R_{ab}\xi'^a-\left(1-\frac{2}{p}\right)\partial_b\delta'\xi'. \tag{4.1}$$

On the other hand, $(\delta'd'\xi')_b=-\nabla^a\nabla_a\xi'_b+\nabla^a\nabla_b\xi'_a$, hence we obtain

$$\nabla^a t(\xi')_{ab}=(S\xi')_b-(\triangle'\xi')_b-\left(1-\frac{2}{p}\right)(d'\delta'\xi')_b. \tag{4.2}$$

LEMMA. *On a compact and orientable LPR-space M, a* $(1,0)$*-form* ξ' *is a* PC_1*-transformation, if and only if,* ξ' *is decomposable and satisfies*

$$\triangle'\xi'+\left(1-\frac{2}{p}\right)d'\delta'\xi'=S\xi', \tag{4.3}$$

provided $r>1$.

Proof. Consider a $(1,0)$-form $u(\xi')$ defined by

$$(u(\xi'))_a=\xi'^b t(\xi')_{ab}.$$

We have

$$\delta'(u(\xi')) = -\xi'^b \nabla^a t(\xi')_{ab} - \nabla^a \xi'^b t(\xi')_{ab}$$

$$= [\triangle'\xi' + \left(1-\frac{2}{p}\right) d'\delta'\xi' - S\xi']_b \xi'^b - \frac{1}{2} < t(\xi'), t(\xi') > .$$

By integration, we have

$$(\triangle'\xi' + \left(1-\frac{2}{p}\right) d'\delta'\xi' - S\xi', \xi') = \frac{1}{2}(t(\xi'), t(\xi')).$$

Hence if ξ' satisfies the equation (4.3), we can conclude $t(\xi')=0$ by using the decomposability of ξ'. Conversely if $t(\xi')=0$, ξ' satisfies the equation (4.3) by the formula (4.2) and is decomposable. This completes the proof.

For a $(0,1)$-form ξ'', we have similar lemma.

From the equation (4.3), we have

$$(4.4) \qquad (d'\xi', d'\xi') + 2\left(1-\frac{1}{p}\right)(\delta'\xi', \xi') = (S\xi', \xi').$$

PROPOSITION 6. *There are no non-trivial (global) 1-parameter groups of PC_1-transformations on a compact and orientable M with the first negative definite Ricci quadratic form.*

If $< S\xi', \xi' > = 0$, *for* $p=0$, we have $< S\xi', \xi' > = 0$, $\delta\xi' = 0$ and $\nabla_a X'^b = 0$. Hence, if the first Ricci quadratic form is negative semi-definite, then a vector field X' on M generating PC_1-transformations of M necessary a parallel field.

PROPOSITION 7. *There are no non-trivial (global) 1-parameter groups of P_1-isometries on a compact and orientable M with the first negative definite Ricci quadratic form.*

PROPOSITION 8. *Let ξ' be an PC_1-transformation on a compact and orientable M. If $\triangle'\xi' = 0$, then $< S\xi', \xi' > = 0$.*

By Lemma, we have the following characterization.

THEOREM 4. *Let ξ be a 1-form such that $\xi = \xi' + \xi''$ on a compact and orientable LPR-space, where ξ' (resp. ξ'') is a $(1,0)$-form (resp. $(0,1)$-form). In order that a 1-form ξ is a PC-transformation, it is necessary and sufficient that ξ is decomposable and, ξ' and ξ'' satisfy*

$$\triangle'\xi' + \left(1-\frac{2}{p}\right) d'\delta'\xi' = S\xi' \text{ and } \triangle''\xi'' + \left(1-\frac{2}{q}\right) d''\delta''\xi'' = S\xi''$$

respectively.

Theorem 4 is a characterization of a PC-transformation by the differential equation.

COROLLARY. *Let* ξ *be a* 1-*form such that* $\xi = \xi' + \xi''$ *on a compact and orientable LPR-space, where* ξ' (*resp.* ξ'') *is a* (1,0)-*form* (*resp.* (0,1)-*form*). *In order that* ξ *is a P-isometry, it is necessary and sufficient that* ξ *is decomposable and* ξ' *and* ξ'' *satisfy* $\triangle'\xi' = S\xi'$, $d'\delta'\xi' = 0$ *and* $\triangle''\xi'' = S''$, $d''\delta''\xi'' = 0$ *respectively. Consequently, if* ξ *is an isometry,* ξ *is* δ-*closed.*

§5. PRODUCT EINSTEIN SPACE

5.1. Let M be a compact and orientable LPR-space. In this section we assume that $p > 2$ and $q > 2$. Let ϕ be a section of a sheaf E defined in §1 on M. We can construct a coherent subsheaf B' (resp.B'') having ϕ as a section which depends upon a finite number of fixed forms in variables of the second (resp. first) type.

Reinhart [9] showed that there exists the Green's operator G' independent from the choice of B'. But G' may be densely defined without being everywhere defined, hence it must in general be unbounded. Thus G' is not always completely continuous.

Let B' be a coherent subsheaf of E. Assume that G' is completely continuous on $\mathfrak{L}_2(B')$. Consider an equation $\triangle'\phi = \lambda\phi$, where $\phi \in \mathfrak{L}_2(B')$. Hence the set of eigenvalues of $\triangle'$ is at most countable and does not converge to bounded value.

Let λ be an eigenvalue of $\triangle'$ and ψ a corresponding eigenform in $\mathfrak{L}_2(B')$. By the equation (1.6), we have $\psi = \psi_1 + \psi_2$, where $\psi_1, \psi_2 \in \mathfrak{L}_2(B')$ and $\delta'\psi_1 = 0$, $d'\psi_2 = 0$.

Let S be a Ricci operator on (1,0)-forms in $\mathfrak{L}_2(B')$. Assume that S is defined on everywhere and that positive definite. Let $\lambda_1(b)$ be the minimal eigenvalue of S at $b \in M$ and define $\lambda_1 = \min_{b \in M} \lambda_1(b)$. We set $\lambda_2 = \lambda_1 \left(2 - \frac{2}{p}\right)^{-1}$.

PROPOSITION 9. *Let* M *be a compact and orientable LPR-space. Let* B' *be an arbitrary coherent subsheaf of* E *on* M *having* ϕ *as a section, where* ϕ *is a section of* E. *Assume that the Green's operator* G' *on* $\mathfrak{L}_2(B')$ *is completely continuous. Let* λ_1 *and* λ_2 *be real numbers defined above. If the Ricci operator* S *on* (1,0)-*forms in* $\mathfrak{L}_2(B')$ *is positive definte, then eigenvalues* λ *and* μ *of* $\triangle'$, *to which correspond* (1,0)-*forms* ψ_1 *and* ψ_2 *respectively, as eigenforms, such that* $\delta'\psi_1 = 0$ *and* $d'\psi_2 = 0$, *are not less than* λ_1 *and* λ_2 *respectively. Moreover, if there exist an eigenform* ψ_1(*resp.* ψ_2) *corresponding eigenvalue* λ *of* $\triangle'$ *such that* $\lambda = \lambda_1$ (*resp.* $\mu = \lambda_2$(, $t(\psi_1) = 0$ (*resp.* $t(\psi_2) = 0$) *holds good.*

We can easily prove this prpposition by the method similar to [7] and the proof is omitted.

5.2. In this paragraph we shall consider an Einstein space. An LPR-space is called a *product Einstein space*, for brevity, a *PE-space*, if the Riemannian metric is Einsteinian. We shall consider a special PE-space with a metric such that $R_{\alpha\beta} = (\lambda_1/2)g_{\alpha\beta}$, where λ_1 is a positive constant. Hence the dimension of a space of d$'$-cohomology class of (1,0)-forms contained in $\mathfrak{L}_2(B')$ is zero, where B' is a coherent subsheaf of E as before. Let $\xi = \xi' + \xi''$ be an arbitrary decomposable 1-form on M where ξ' (resp. ξ'') is a (1,0)-form (resp. (0,1)-form) in $\mathfrak{L}_2(B')$ (resp. $\mathfrak{L}_2(B'')$). In this case, we can take a subsheaves B' and B'' arbitrary. On the other hand, ξ' is written by $\xi' = \zeta' + \omega'$, where $\delta'\zeta' = 0$, $d'\omega' = 0$. If ξ' defines an PC$_1$-transformation, we have $\triangle'(\zeta'+\omega') + \left(1 - \frac{2}{p}\right) d'\delta'\omega' = \lambda_1(\zeta'+\omega')$, i.e., $\delta' d'\zeta' + \left(2 - \frac{2}{p}\right) d'\delta'\omega' = \lambda_1(\zeta' + \omega')$. Hence we get $\triangle'\zeta' = \lambda_1\zeta'$ and $\triangle'\omega' = \lambda_2'\omega'$, where $\lambda_2' = \lambda_1\left(2 - \frac{2}{p}\right)^{-1}$. The last equations mean that λ_1(resp.λ_2') is an eigenvalue of $\triangle'$ and ζ' (resp. ω') is a corresponding eigenform. Therefore, ζ' is a P$_1$-isomitry and ω' is a PC$_1$-transformation with $d'\omega' = 0$. For a (0,1)-form ξ'' in $\mathfrak{L}_2(B'')$, similarly we have $\triangle''\zeta'' = \lambda\zeta''$ and $\triangle''\omega'' = \lambda_2''\omega''$, where $\xi'' = \zeta'' + \omega''$, $\delta''\zeta'' = 0$, $d''\omega'' = 0$ and $\lambda_2'' = \lambda_1\left(2 - \frac{2}{q}\right)^{-1}$. Therefore ζ'' corresponds to an eigenvalue of $\triangle''$ and ζ'' is a P$_2$-isometry and ω'' corresponds to an eigenvalue λ_2'' of $\triangle''$ and ω'' is a PC$_2$-transformation. If we set $\zeta = \zeta' + \zeta''$ and $\omega = \omega' + \omega''$, ζ is a P-isometry and ω is a PC-transformation. Then we have the following proposition.

PROPOSITION 10. *Let M be a compact and orientable PE-space such that $R_{\alpha\beta} = (\lambda_1/2)g_{\alpha\beta}$, $\lambda_1 > 0$ and $p > 2$, $b > 2$. If a 1-form ξ on M is a PC-transformation, ξ is decomposed into $\xi = \zeta + \omega$ and ζ is a P-isometry and ω is a PC-transformation, where ζ and ω are 1-forms satisfying $\delta\zeta = d\,\omega = 0$.*

In general it is known that all the infinitesimal automorphisms of a G-structure admit a structure of Lie algebra. We shall consider a structure of Lie algebra in our case.

It is clear that the Poisson bracket $[\xi, \xi]$ of two decomposable 1-forms ξ is and ξ also decomposable.

Let L' (resp. L'') be a Lie algebra of (1,0)-forms (resp. (0,1)-forms) generating PC$_1$-transformation (resp. PC$_2$-transformations). Let L be a Lie algebra of 1-forms generating PC-transformations. Hence we have $L \approx L' \times L''$. Let L_1 be a Lie algebra of decomposable 1-forms ζ satisfying $\triangle\zeta = \lambda_1\zeta$ and $\delta'\zeta' = \delta''\zeta'' = 0$. Let L'_1 (resp. L''_1) be a Lie algebra of decomposable (1,0)-froms ζ' (resp. (0,1)-forms ζ'') satisfying $\triangle'\zeta' = \lambda_1\zeta'$ and $\delta'\zeta' = 0$ (resp. $\triangle''\zeta'' = \lambda_1\zeta''$ and $\delta''\zeta'' = 0$). We have $L_1 = L'_1 \times L''_1$. Let L'_2(resp. L''_2) be a space of decomposable (1,0)-forms ω' (resp. (0,1)-forms ω'') satisfying $\triangle'\omega' = \lambda'_2\omega'$ and $d'\omega' = 0$ (resp. $\triangle''\omega'' = \lambda''_2\omega''$ and $d''\omega'' = 0$). If we set $L_2 = L'_2 + L''_2$, we have $L = L_1 \oplus L_2$, $[L_1, L_1] \subset L_1$, $[L_1, L_2] \subset L_2$, $[L_2, L_2] \subset L_1$ by Theorem of p. 138 of [7].

§6. REMARKS ON DECOMPOSABLE 1-FORMS

Let M be an LPR-space. The almost-product structure on M is characterized by a tensor field $\varphi = (\varphi^{\alpha}_{\beta})$ defined by $(Jv)^{\alpha} = \varphi^{\alpha}_{\beta} v^{\beta}$. We can easily show that a 1-form ξ is decomposable if and only if $\pounds(\xi)\varphi = 0$. Now we construct a tensor $a(\xi) = (a_{\alpha\beta}(\xi))$, where $a_{\alpha\beta}(\xi) = \frac{1}{2} (\pounds(\xi)\varphi)^{\gamma}_{\alpha} \varphi_{\gamma\beta}$ and $\varphi_{\beta\gamma} = \varphi^{\lambda}_{\gamma} g_{\lambda\beta}$. Hence components of a tensor $a(\xi)$ are given by

$$a_{ab}(\xi) = 0,\ a_{aj}(\xi) = \nabla_a \xi_j,\ a_{ib}(\xi) = \nabla_i \xi_b,\ a_{ij}(\xi) = 0\,.$$

Therefore $a_{\alpha\beta}(\xi) = 0$ is the necessary and sufficient condition that ξ is decomposable.

If M is compact and orientable, we have

$$(a(\xi'), a(\xi')) = (\xi', \triangle''\xi'),\ (a(\xi''), a(\xi'')) = (\xi'', \triangle'\xi''),$$

where $\xi = \xi' + \xi''$ and ξ' (resp. ξ'') is a $(1,0)$-form (resp. $(0,1)$-form). On the other hand, we have

$$(\triangle''\xi')^{a} = \nabla^{i}\nabla_{i}\xi'^{a},\ (\triangle'\xi'')^{i} = \nabla^{a}\nabla_{a}\xi''^{i}.$$

Hence we have the following conclusion.

PROPOSITION 11. *On a compact and orientable LPR-space, in order that a 1-form ξ such that $\xi = \xi' + \xi''$, where ξ' (resp, ξ'') is a $(1,0)$-form (resp. $(0,1)$-form), is decomposable it is necessary and sufficient that* $\triangle''\xi' = \triangle'\xi'' = 0$.

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IWATE UNIVERSITY, MORIOKA.