Note on the isomorphism problem of N-semigroups

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In the previous papers [3], [4], the author has discussed the isomorphism problem of N-semigroups, that is, commutative, nonpotent, cancellative, archimedean semigroups, using the Tamura's representation. Lately, J. Higgins [2] gave an isomorphism theorem for finitely generated N-semigroups, which depended on the canonical representation. The author, in [5], has announced an isomorphism theorem in general case without proof, which generalize the above Higgins' result. In this paper, we shall prove this theorem. The notation and terminology are based on [5].

Let S and S' be N-semigroups and let $S(G:I)$ and $S(G':I')$ be arbitrary Tamura's representations [6] for S and S' respectively. Suppose that S is isomorphic upon S' under φ. Let e and e' be the identities of G and G' respectively and put $(0,e)\varphi=(n',a')$ and $(0,e')\varphi^{-1}=(n,a)$. Since, then, $(0,e')=(n,a)\varphi=((0,a)(0,e)^n)\varphi$, it follows that $nn'=0$. Hence we may suppose that $n'=0$ without loss of the generality, so hereafter we shall discuss on this supposition.

Let $G_{(0,e)}, G'_{(0,a')}$ and $I_{(0,e)}, I'_{(0,a')}$ be the structure groups with respect to $(0,e)\in S$, $(0,a')\in S'$ and their corresponding index functions respectively, and let ρ,ρ' be congruences on S,S' such that $G_{(0,e)}=S/\rho$, $G'_{(0,a')}=S'/\rho'$.

Consider a mapping τ: $(r,x)\rho\to((r,x)\varphi)\rho'$, where $(r,x)\rho$, $((r,x)\varphi)\rho'$ are the equivalence classes modulo ρ,ρ' containing $(r,x)\in S$, $(r,x)\varphi\in S'$ respectively. Then it is easily shown that τ is an isomorphism of $G_{(0,e)}$ onto $G'_{(0,a')}$.

Let $I_{(0,e)}((r,x)\rho,(r',x')\rho)=p$, where (r,x) and (r',x') are primes of S respecting to $(0,e)$. Since, then, there exists a prime (s,y) of S respecting to $(0,e)$ such that $(r,x)(r',x')=(0,e)^p(s,y)$, it follows that $(r,x)\varphi\cdot(r',x')\varphi=(0,a')^p\cdot(s,y)\varphi$. Since $(r,x)\varphi$, $(r',x')\varphi$, and $(s,y)\varphi$ become primes of S' respecting to $(0,a')$, it follows that $I'_{(0,a')}(((r,x)\varphi)\rho',((r',x')\varphi)\rho')=p$, hence

$$I_{(0,e)}((r,x)\rho,\ (r',x')\rho)=I'_{(0,a')}(((r,x)\rho)\tau,\ ((r',x')\rho)\tau)\,.$$

Hence $G_{(0,e)}$ with $I_{(0,e)}$ is equivalent to $G'_{(0,a')}$ with $I'_{(0,a')}$, which shall be denoted by $(G_{(0,e)},I_{(0,e)})\sim(G'_{(0,a')},I'_{(0,a')})$. Clearly $(G,I)\sim(G_{(0,e)},I_{(0,e)})$, hence we get $(G,I)\sim(G'_{(0,a')},I'_{(0,a')})$.

Conversely, suppose that for given N-semigroups $S=S(G:I)$, $S'=S(G':I')$ there exists $(0,a')\in S'$ such that $(G,I)\sim(G'_{(0,a')},I'_{(0,a')})$. Since $(G,I)\sim(G_{(0,e)},I_{(0,e)})$, it follows $(G_{(0,e)},I_{(0,e)})\sim(G'_{(0,a')},I'_{(0,a')})$, hence there exists an isomorphism τ of $G_{(0,e)}$ onto $G'_{(0,a')}$ and $I_{(0,e)}(\xi,\eta)=I'_{(0,a')}(\xi\tau,\eta\tau)$ for every $\xi,\eta\in G_{(0,e)}$. Let $S/\rho=G_{(0,e)}$, $S'/\rho'=G'_{(0,a')}$ and define a mapping φ as follows:

$$\varphi:\ (n,(r,x)\rho)\to(n,((r,x)\rho)\tau)\,,$$

where $(r,x)\rho$ is the equivalence class modulo ρ containing (r,x). Then φ is a one-to-one mapping of $S(G_{(0,e)}:I_{(0,e)})$ onto $S(G'_{(0,a')}:I'_{(0,a')})$.

And
$$\begin{aligned}&((n,(r,x)\rho)\cdot(n',(r',x')\rho))\varphi\\ &=(n+n'+I_{(0,e)}((r,x)\rho,\ (r',x')\rho),\ (((r,x)(r',x'))\rho)\tau)\\ &=(n+n'+I'_{(0,a')}(((r,x)\rho)\tau,\ ((r',x')\rho)\tau),\ (((r,x)(r',x'))\rho)\tau)\\ &=(n,((r,x)\rho)\tau)(n',((r',x')\rho)\tau)\\ &=(n,(r,x)\rho)\varphi\cdot(n',(r',x')\rho)\varphi.\end{aligned}$$

Hence $S(G_{(0,e)}:I_{(0,e)})$ is isomorphic upon $S(G'_{(0,a')}:I'_{(0,a')})$ under φ. Therefore

$$S(G:I)\cong S(G_{(0,e)}:I_{(0,e)})\cong S(G'_{(0,a')}:I'_{(0,a')})\cong S(G':I').$$

Thus we obtain the following theorem :

THEOREM. *Let S and S' be N-semigroups and let $S(G:I)$ and $S(G':I')$ be Tamura's representations for S and S' respectively. Then S is isomorphic upon S' if and only if there exists an element $(0,a')\in S(G':I')$ such that $(G,I)\sim(G'_{(0,a')},\ I'_{(0,a')})$.*

Let S and S' be finitely generated N-semigroups and let $S(G:I)$ and $S(G':I')$ be canonical representations for S and S' respectively. Since, then, $(0,e)$ is a normal standard of S and any isomorphism of S onto S' preserves normal standard elements, so the elemtent $(0,a')$ in the above theorem becomes a normal standard of S'. Hence we get the following corollary, due to J. Higgins [2].

COROLLARY. *Let S and S' be finitely generated N-semigroups and let $S(G:I)$ and $S(G':I')$ be canonical representations for S and S' respectively. Then S is isomorphic upon S' if and only if there exists a normal standard element $(0,a')\in S(G':I')$ such that $(G,I)\sim(G'_{(0,a')},\ I'_{(0,a')})$.*

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