On a Period Analysis for Musical sound of Certain Musical Instrumet of Percussion

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In a musical sound of striking musical instrument, for example, Korean Stone musical instruments at Confucius Shrine in Seoul, though it is beautiful, we can not always say that in all components of a musical sound their frequencies have multiple integral numbers of their least frequency.

If we consider such a case that logarithmic decrements of component waves are different and some of them have an increasing amplitude, we can apply Fourier's[1] Analysis for only a periodic wave with constant state.

In the experiment, as ordinarily a sound photographic image is given by the following expression:-

$$U=\sum ae^{-kt} \sin\frac{2\pi}{T}(t+\gamma) \qquad (1)$$

For (1) we can apply an Analysis of E.[2] T. Whittaker and G. Robinson in the special case of $K=0$. I.[3] Aoki used their method and tried to make acoustic spectrums with analysed sine wave groups. K.[4] Takahashi tried the another period analysis for the different things from sound.

O.[5] F. Socia's mechanical method which is applied for aperiodic sound waves stands the another stand point with us. We will propose a method of period Analysis applicable for sound wave whose components have $K\neq 0$.

We draw two axes on the properly enlarged photographic image of sound wave, one is parallel to the time axis of wave and the other is perpendicular to it, then denote the ordinates corresponding to the end points of the properly small equidistantly divided time intervals, as follows:-

$V_0,\ V_1,\ V_2,\ V_3, \cdots\cdots\cdots,\ V_i, \cdots\cdots\cdots\cdots$

We will use u_i for the standard deviation of $V_i's$ that is

$$u_i = V_i - V_m \qquad (i = 0, 1, 2, 3, \cdots\cdots\cdots\cdots)$$

Where V_m = the arithmetic mean of $V_i's$.

and we make a following arrangement with $(2n+1)$ lines and p columns of successive $u_i's$:-

u_0	u_1	u_2	〃	〃	u_i	〃	〃	〃	u_{p-2}	u_{p-1}
u_p	u_{p+1}	u_{p+2}	〃	〃	u_{p+i}	〃	〃	〃	u_{2p-2}	u_{2p-1}
〃	〃	〃	〃	〃	〃	〃	〃	〃	〃	〃
〃	〃	〃	〃	〃	〃	〃	〃	〃	〃	〃
u_{np}	u_{np+1}	u_{np+2}	〃	〃	n_{np+i}	〃	〃	〃	$u_{(n+1)p-2}$	$u_{(n+1)p-1}$
〃	〃	〃	〃	〃	〃	〃	〃	〃	〃	〃
〃	〃	〃	〃	〃	〃	〃	〃	〃	〃	〃
〃	〃	〃	〃	〃	〃	〃	〃	〃	〃	〃
u_{2np}	u_{2np+1}	u_{2np+2}	〃	〃	u_{2np+i}	〃	〃	〃	$u_{(2n+1)p-2}$	$n_{(2n+1)p-1}$

(2)

where u_{np+i} = central term in the arrangement.

(1) 佐藤孝二:科学 7(1937), 364. Kalähne: Grundräge der math-phys. Austik 1.kap.2

(2) E. T. Whittaker and G. Robinson: Calculus of observation. p.p. 343-360 (1926)

(3) I. Aoki: memoires, coll. of Scie. Kyoto, 14, 213-218(1931)

(4) 高橋浩一: 理化学研究所彙報 第十四輯第四號 PP. 225—271(昭和十年四月)

(5) C. F. Sacia: J. opt. soc. of. amer., vol. 9, 1929, P. 487.

In (2) ☆ 以下本文中 Ki は K_i を示し，K1 は K_1 を示す。(印刷の都合上)

$$u_t = a_1 e^{-K_1 t} \sin\frac{2\pi}{T_1}(t+\gamma_1) + \sum_{i=2}^{m} a_i e^{-K_i t} \sin\frac{2\pi}{T_i}(t+\gamma_i) \qquad (3)$$

where $\begin{cases} a_1 e^{-K_1 t} \sin\frac{2\pi}{T_1}(t+\gamma_1) = a \text{ Wave component with period } T_1. \\ m = a \text{ number of wave components contianed the Sound.} \end{cases}$

Now we use following symbols,-

$$U_t \equiv \sum a_1 e^{-K_1(t+\alpha p\nu)} \sin(t+\gamma_1+\alpha p\nu) + \sum_{\alpha=0}^{2n} \sum_{i=2}^{m} a_i e^{-Ki(t+\alpha p\nu)} Sin \frac{2\pi}{T_i}(t+\gamma_i+\alpha p\nu) \qquad (4)$$

where ν=equidevided time interval.

and

$$M_t^2 \equiv \frac{1}{(2n+1)^2} U_t^2 \qquad (5)$$

The magnetude of ν must be taken so small as the following three approximations are satisfied .

$$\int_0^p u_{ip+t}\, dt \fallingdotseq \nu \sum_{i=0}^{p-1} u_{ip+t}$$

$$\int_0^p U_t^2 dt \fallingdotseq \nu \sum_{i=0}^{p-1} U_t^2$$

$$\int_0^p M_t^2\, dt \fallingdotseq \nu \sum_{i=0}^{p-1} M_t^2$$

Case (1)

A case of no integral multiple ratio between any two frequencies of components in a sound.

If there is no integral multiple ratio between any two periods of components in a sound, the following two assumptions are brought, that is, if we have no componet wave whose period T_i has like $(T_1, T_i) = T_1$ *in* $\sum_{i=2}^{m} a_i e^{-Kit} sin \frac{2\pi}{T_i}(t+\gamma_i)$,

we have:-

(i) There is no correlation between $\sum_{\alpha=0}^{2n} a_1 e^{-K_1(t+\alpha p\nu)} sin \frac{2\pi}{T_1}(t+\gamma_1+\alpha p\nu)$

and $\sum_{\alpha=0}^{2n} \sum_{i=2}^{m} a_i e^{-Ki(t+\alpha p\nu)} sin \frac{2\pi}{T_i}(t+\gamma_i+\alpha p\nu)$

(ii) There is no correlation between $\sum_{i=2}^{m} a_i e^{-Ki(t+\alpha p\nu)} sin \frac{2\pi}{T_i}(t+\gamma_i+\alpha' p\nu)$

and $\sum_{\alpha=0}^{\alpha'-1} \sum_{i=2}^{m} a_i e^{-Ki(t+\alpha\nu)} sin \frac{2\pi}{T_i}(t+\gamma_i+\alpha p\nu) + \sum_{\alpha=\alpha'+1}^{2n} \sum_{i=2}^{m} a_i e^{-Ki(t+\alpha p\nu)} sin \frac{2\pi}{T}(t+\gamma_i+\alpha p\nu)$

where $\alpha' =$ any number of α

From (i)

$$\frac{1}{p\nu}\int_0^{p\nu} M_t^2 dt = \frac{1}{(2n+1)^2 p\nu}\int_0^{p}\left[\left\{\sum a_1 e^{-K_1(t+\alpha p\nu)} sin\ \frac{2\pi}{T_1}(t+\gamma_1+\alpha p\nu)\right\}^2 + \left\{\sum_{\alpha=0}^{2n}\sum_{i=2}^{m} a_i e^{-Ki(t+\alpha p\nu)} sin\ \frac{2\pi}{T_i}(t+\gamma_i+\alpha p\nu)\right\}^2\right]dt$$

From (ii)

$$\int_0^{p\nu}\left[\sum_{\alpha=0}^{2n}\left\{\sum_{i=2}^{m} a_i e^{-Ki(t+\alpha p\nu)} sin\ \frac{2\pi}{T_i}(t+\gamma_i+\alpha p\nu)\right\}^2\right]dt$$

$$=\int_0^{p\nu}\left[\sum_{\alpha=0}^{2n}\left\{\sum_{i=2}^{m} a_i e^{-Ki(t+\alpha p\nu)} sin\frac{2\pi}{T_i}(t+\gamma_i+\alpha p\nu)\right\}^2\right]dt$$

$$=\int_0^{d\nu}\left\{\sum_{i=2}^{m} a_i e^{-Ki(t+\alpha p\nu)} Sin\frac{2\pi}{T_i}(t+\gamma_i+\alpha p\nu)\right)$$

$$=(2m+1)p\nu\sigma_{ai}^2 \qquad (6)$$

where $\sigma_{ai} =$ standard deviation of $\sum a_i e^{-Kit} Sin\ \frac{2\pi}{T_i}(t+\gamma_i)$

Now, with respect to $\sum_{\alpha=0}^{2n} a_1 e^{-K_1(t+\alpha p\nu)} Sin\ \frac{2\pi}{T_1}(t+\gamma_1+\alpha p\nu)$, $K_i\ \alpha p\nu$

be so small that we can assume the following expression:-

$$\left.\begin{aligned} e^{-K_1\alpha p\nu} &\fallingdotseq 1-K_1\alpha p\nu \\ e^{K_1\alpha p\nu} &\fallingdotseq 1+K_1\alpha p\nu \end{aligned}\right\}\quad 0\leqq\alpha<2n+1$$

then

$$a_1 e^{-K_1(t+(n+i)p\nu)} Sin\frac{2\pi}{T_1}(t+\gamma_1(n+i)p\nu)$$

$$=a_1(1-K_1 ip\nu)e^{-K_1(t+np\nu)} Sin\frac{2\pi}{T_1}(t+\gamma_1+(n+i)p\nu)$$

and

$$a_1 e^{-K_1(t+(n-i)p\nu)} Sin\ \frac{2\pi}{T_1}(t+\gamma_1+(n-i)p\nu)$$

$$=a_1(1+K_1 ip\nu)e^{-K_1(t+np\nu)} Sin\ \frac{2\pi}{T_1}(t+\gamma_1+(n-i)p\nu)$$

$\therefore$

$$\sum_{\alpha=0}^{2n} a_1 e^{-K_1(t+\alpha p\nu)} Sin \frac{2\pi}{T_1}(t+\gamma_1+\alpha p\nu)$$

$$=a_1 e^{-K_1(t+\alpha p\nu)} Sin \frac{2\pi}{T_1}(t+\gamma_1+np\nu)\left(\frac{Sin\frac{(2n+1)\pi p\nu}{T_1}}{Sin\frac{\pi p\nu}{T_1}}\right)$$

$$-2K_1 p\nu a_1 e^{-K_1(t+np\nu)} Cos\frac{2\pi}{T_1}(t+\gamma_1+np\nu)\sum_{N=1}^{n} NSin\frac{2N\pi}{T_1}p\nu$$

consequently we have

$$\frac{1}{(2n+1)^2 p\nu}\int_0^{p\nu}\left\{\sum_{\alpha=0}^{2n} a_1 e^{-K_1(t+\alpha p\nu)} Sin \frac{2\pi}{T_1}(t+\gamma_1+\alpha p\nu)\right\}^2 dt$$

$$=\frac{a_1^2 e^{-2K_1 np\nu}}{2(2n+1)^2 p\nu}\left(\frac{1-e^{-2K_1 p\nu}}{2K_1}\right)\left\{\left(\frac{Sin\frac{(2n+1)\pi p\nu}{T_1}}{Sin\frac{\pi p\nu}{T}}\right)^2+4K_1^2p^2\nu^2\left(\sum_{N=1}^{n}Nsin\frac{2N\pi p\nu}{T_1}\right)^2\right\}$$

$$+\frac{a_1^2 e^{-2K_1 np\nu}}{2(2n+1)^2 p\nu}\left\{-\left(\frac{Sin\frac{(2n+1)\pi p\nu}{T_1}}{Sin\frac{\pi p\nu}{T_1}}\right)^2\right.$$

$$\left.+4K_1^2p^2\nu^2\left(\Sigma NSin\frac{2N\pi p\nu}{T_1}\right)^2\right\}\frac{\left[e^{-2K_1 t}Sin\frac{4\pi}{T_1}\left(t+\gamma_1+np\nu+\frac{T_1\delta}{4\pi}\right)\right]_0^{p\nu}}{\frac{4\pi}{T_1}\sqrt{1+\frac{K_1^2\,T_1^2}{4\pi^2}}}$$

$$-\frac{a_1^2 e^{-2K_1 np\nu}}{(2n+1)^2 p\nu}K_1^2p^2\nu^2\left(\frac{Sin\frac{(2n+1)\pi p\nu}{T_1}}{Sin\frac{\pi p\nu}{T_1}}\right)\left(\sum_{N=1}^{n}NSin\frac{2N\pi p\nu}{T_1}\right)$$

$$\times\frac{\left[e^{-2K_1 t}Cos\frac{4\pi}{T_1}\left(t+\gamma_1+np\nu+\frac{T_1\delta}{4\pi}\right)\right]_0^{p\nu}}{\frac{4\pi}{T_1}\sqrt{1+\frac{K_1^2\,T_1^2}{4\pi^2}}} \qquad (7)$$

where $\tan\delta=-\frac{K_1\,T_1}{2\pi}$

But

$$\left|\frac{\left[e^{-2K_1 t}Sin\frac{4\pi}{T_1}\left(t+\gamma_1+np\nu+\frac{T_1\delta}{4\pi}\right)\right]_0^{p\nu}}{\frac{4\pi}{T_1}\sqrt{1+\frac{K_1^2\,T_1^2}{4\pi^2}}}\right|<\frac{1}{\frac{2\pi}{T_1}\sqrt{1+\frac{K_1^2\,T_1^2}{4\pi^2}}}$$

In our experiment, we have $\frac{1}{T_1} > 100$

So that

$$\left| \frac{\left[e^{-2K_1 t} Sin \frac{4\pi}{T_1} (t + \gamma_1 + np\nu + \frac{T_1 \delta}{4\pi}) \right]_0^{p\nu}}{\frac{4\pi}{T_1} \sqrt{1 + \frac{K_1^2 T_1^2}{4\pi^2}}} \right| < \frac{1}{600 \sqrt{1 + \frac{K_1^2 T_1^2}{4\pi^2}}} < \frac{1}{600}$$

similarly

$$\left| \frac{\left[e^{-2K_1 t} cos \frac{4\pi}{T_1} (t + \gamma_1 + np\nu + \frac{T_1 \delta}{4\pi}) \right]_0^{p\nu}}{\frac{4\pi}{T_1} \sqrt{1 + \frac{K_1^2 T_1^2}{4\pi^2}}} \right| < \frac{1}{600}$$

Agian

$$\left(\frac{Sin \frac{(2n+1)\pi p\nu}{T_1}}{Sin \frac{\pi p\nu}{T_1}} \right) \leqq 2n+1$$

and

$$\sum_{N=1}^{n} N \; Sin \frac{2N\pi p\nu}{T_1} < \frac{n(n+1)}{2}$$

Now

$$e^{-K_1(n+1)p\nu} = 1 - K_1(n+1)p\nu > 0$$

$$\therefore \quad K_1 p\nu < \frac{1}{n+1}$$

Consequeutly, in calculation of (7) we can neglect terms which contains

$$\frac{\left[e^{-2K_1 t} Sin \frac{4\pi}{T_1} (t + \gamma_1 np\nu + \frac{T_1 \delta}{4\pi}) \right]_0^{p\nu}}{\frac{4\pi}{T_1} \sqrt{1 + \frac{K_1^2 T_1^2}{4\pi^2}}} \quad or \quad \frac{\left[e^{-2K_1 t} cos \frac{4\pi}{T_1} (t + \gamma_1 + np\nu + \frac{T_1 \delta}{4\pi}) \right]_0^{p\nu}}{\frac{4\pi}{T_1} \sqrt{1 + \frac{K_1^2 T_1^2}{4\pi^2}}}.$$

in each value of p.

Therefore (7) can be expressed followingly,—

$$\frac{1}{(2n+1)^2 p\nu} \int_0^{p\nu} \left\{ \sum_{\alpha=0}^{2n} a_1 e^{-K_1(t+\alpha p\nu)} Sin \frac{2\pi}{T_1} (t + \gamma_1 + \alpha p\nu) \right\}^2 dt$$

$$\doteqdot \frac{a_1^2 e^{-K_1 n p\nu}}{2(2n+1)^2}\left(\frac{1-e^{-2K_1 n p\nu}}{2K_1}\right)\left\{\left(\frac{Sin\frac{(2n+1)\pi p\nu}{T_1}}{Sin\frac{\pi p\nu}{T_1}}\right)^2 + 4K_1^2 p^2 \nu^2\left(\sum N sin\frac{2N\pi p\nu}{T_1}\right)^2\right\} \quad (8)$$

$\therefore$

$$\frac{1}{p\nu}\int_0^{p\nu}\Sigma_t^2\,dt \doteqdot \frac{a_1^2 e^{-2K_1 n p\nu}}{2(2n+1)^2}\left\{\left(\frac{Sin\frac{(2n+1)\pi p\nu}{T_1}}{Sin\frac{\pi p\nu}{T^1}}\right)^2 + 4K_1^2 p^2 \nu^2\left(\sum N sin\frac{2N\pi p\nu}{T_1}\right)^2\right\}$$

$$+\frac{\sigma_{ai}^2}{2n+1}$$

$$\equiv \Sigma_m^2 \quad (9)$$

where $\Sigma_m =$ standard deviation of M_t's

But

$$4K_1^2 p^2 \nu^2\left(\sum N sin\frac{2N\pi p\nu}{T_1}\right)^2 < 1 \qquad for \quad p\nu \neq T_1,$$

$$= 2n+1 \quad for \quad p\nu = T_1.$$

and

$$\left(\frac{Sin\frac{(2n+1)\pi p\nu}{T_1}}{Sin\frac{\pi p\nu}{T^1}}\right)^2 = fairy\ Small\ value \qquad for \quad p\nu \neq T_1,$$

$$= (2n+1)^2 \qquad for \quad p\nu = T_1.$$

Therefore

$$\text{max. Value of } \Sigma_m^2 \doteqdot \frac{a_1 e^{-2K_1 n T_1}}{2} + \frac{\sigma_{ai}^2}{2n+1} \quad for \quad p\nu = T_1. \quad (10)$$

More over the standard deviation of u's is given by following:-

$$\sigma^2 = \frac{1}{(2n+1)p\nu}\int_0^{(2n+1)p\nu}\left\{a_1 e^{-K_1 t} Sin\frac{2\pi}{T_1}(t+\gamma_1) + \left(\sum a_i e^{-K_i t} Sin\frac{2\pi}{T_i}(t+\gamma_i)\right)\right\}^2 dt$$

$$= \frac{1}{(2n+1)p\nu}\int_0^{(2n+1)p\nu} a_1^2 e^{-2K_1 t}\left(Sin\frac{2\pi}{T_1}(t+\gamma_1)\right)^2 dt + \sigma_{ai}^2$$

$$\doteqdot \frac{a_1^2}{2(2n+1)p\nu}\,\frac{1-e^{-2K_1(2n+1)p\nu}}{2K_1} + \sigma_{ai}^2 \quad (11)$$

But

$$\frac{1-e^{-2K_1(2n+1)p\nu}}{2K_1(2n+1)p\nu}=\frac{e^{-2K_1 np\nu}\left\{(e^{2K_1 np\nu}-1)+(1+e^{-2K_1(n+1)p\nu})\right\}}{2K_1(2n+1)p\nu}$$

$$=\frac{e^{-2K_1 np\nu}\left\{2K_1 np\nu+2K_1(n+1)p\nu\right\}}{2K_1(2n+1)p\nu}$$

$$=e^{-2K_1 np\nu}$$

So (11) becomes the next,

$$\sigma^2=\frac{a_1^2 e^{-2K_1 np\nu}}{2}+\sigma_{ai}^2 \tag{12}$$

we call the next expression "Periodgram Fquation,"

$$\eta^2=\frac{\sum_{m}^2}{\sigma^2} \tag{13}$$

(13) becomes as follows:-

$$\eta^2=\frac{\dfrac{a_1^2 e^{-2K_1 np\nu}}{2(2n+1)^2}\left\{\left(\dfrac{Sin\dfrac{(2n+1)\pi p\nu}{T_1}}{Sin\dfrac{\pi p\nu}{T_1}}\right)^2+4K_1^2p^2\nu^2\left(\sum N sin\dfrac{2N\pi p\nu}{T_1}\right)^2\right\}+\dfrac{\sigma_{ai}^2}{2n+1}}{\dfrac{a_1^2 e^{-2K_1 np\nu}}{2}+\sigma_{ai}^2} \tag{14}$$

and we get the maximum value of η^2 for a wave whose amplitude is a_1, as follows;-

$$\eta^2_{max}=\frac{\dfrac{a_1^2 e^{-2K_1 np\nu}}{2}+\dfrac{\sigma_{ai}^2}{2n+1}}{a_1^2 e^{-2K_1 np\nu}+\sigma_{ai}^2} \qquad for \quad p\nu=T_1. \tag{15}$$

Expressions (14) and (15) be applicable for any component wave so if we draw a graph by using abscissa p, and ordinate η^2, its peak give us point to be reesearched for a wave component of period T_1.

And amplitude a_1 is studied by the following treatments

$$a_1 e^{-K_1 nT_1}=\sqrt{\frac{2n+1}{n}\left(\sum_{T_1 m}^2-\frac{\sigma_{T_1}^2}{2n+1}\right)} \tag{16}$$

$$where\begin{cases}\Sigma_{T_1 m} = Standard\ deviation\ of\ M's\quad for\quad p\nu = T_1, \\ \sigma_{T_1} \quad = Standard\ deviation\ of\ u's\quad for\quad p\nu = T_1,\end{cases}$$

In (16) if we get $\Sigma^2_{T_1 m} - \frac{\sigma^2_{T_1}}{2n+1} \leqq 0$, a_1 is no existence.

We do the same treatment for the arrangement of u's from the second line to the $(2n+2)$ th line, and we have

$$a_1 e^{-K_1 (n+1) T_1} = \sqrt{\frac{2n+1}{n}\left(\Sigma'^2_{T_1 m} - \frac{\sigma'^2_{T_1}}{2n+1}\right)} \tag{17}$$

$$where\begin{cases}\Sigma'^2_{T_1 m} = standard\ deviation\ of\ M's\quad for\quad p\nu = T_1, \\ \sigma'^2_{T_1} = standard\ deviation\ of\ u's\quad for\quad p\nu = T_1.\end{cases}$$

From (16) and (17), logarithmic decrement and amplitude a_1 are given as follows:-

$$-\frac{K_1 T_1}{2} = \frac{1}{4}\left\{ log\left(\Sigma'^2_{T_1 m} - \frac{\sigma_{T_1}'^2}{2n+1}\right) - log\left(\Sigma^2_{T_1 m} - \frac{\sigma^2_{T_1}}{2n+1}\right)\right\} \tag{18}$$

and

$$a_1 = a_1 e^{-K_1 n T_1} \cdot e^{K_1 n T_1}$$

$$= \sqrt{\frac{2n+1}{n}\left(\Sigma^2_{T_1 m} - \frac{\sigma^2_{T_1}}{2n+1}\right)}\left[1 + \left\{\frac{n}{2}\left(log\left(\Sigma^2_{T_1 m} - \frac{\sigma_{T_1 m}'^2}{2n+1}\right) - log\left(\Sigma'^2_{T_1 m} - \frac{\sigma_{T_1}'^2}{2n+1}\right)\right)\right\}\right] \tag{19}$$

Case (11)

When C is certain integer, we will consider the case in which a sound wave contains a wave component whose period CT_1, that is, CT_1 is C times the peried T_1.

In this case we have

$$U_t = \sum_{\alpha=0}^{2n}\left\{a_1 e^{-K_1 (t+\alpha p\nu)} Sin\frac{2\pi}{T^1}(t+\gamma_1+\alpha p\nu) + a_1' e^{-K_1' (t+\alpha p\nu)} Sin\frac{2\pi}{CT_1}(t+\gamma_1'+\alpha p\nu)\right\}$$

$$+ \sum_{\alpha=0}^{2n}\sum_{i=3}^{m} a_i e^{-Ki(t+\alpha p\nu)} Sin\frac{2\pi}{T_1}(t+\gamma_1+\alpha p\nu) \tag{20}$$

$$M_t^2 = \frac{1}{(2n+1)^2}\left\{\sum_{\alpha=0}^{2n} a_1 e^{-K_1 (t+\alpha p\nu)} Sin\frac{2\pi}{T_1}(t+\gamma_1+\alpha p\nu)\right\}^2$$

$$+\frac{1}{(2n+1)^2}\left\{\sum_{\alpha=0}^{2n} a_1' e^{-K_1'(t+\alpha p\nu)} Sin\frac{2\pi}{CT_1}(t+\gamma_1'+\alpha p\nu)\right\}^2$$

$$+\frac{1}{(2n+1)^2}\left\{\sum_{\alpha=0}^{2n}\sum_{i=3}^{m} a_i e^{-K_i'(t+\alpha p\nu)} Sin\frac{2\pi}{T_i}(t+\gamma_i+\alpha p\nu)\right\}^2$$

$$+\frac{a_1 a_1' e^{-(K_1+K_1')(t+np\nu)}}{(2n+1)}\left\{\left(\frac{Sin\frac{(2n+1)\pi p\nu}{T_1}}{Sin\frac{\pi p\nu}{T_1}}\right)\left(\frac{Sin\frac{(2n+1)\pi p\nu}{CT_1}}{Sin\frac{\pi p\nu}{CT_1}}\right)\times\right.$$

$$\frac{cos\frac{2\pi}{T_1}\left\{(1+\frac{1}{C})t+(\gamma_1+\frac{\gamma_1'}{C})\right\}-Cos\frac{2\pi}{T_1}\left\{(1-\frac{1}{C})t+(\gamma_1-\frac{\gamma_1'}{C})\right\}}{2}$$

$$+2K_1K_1'p\nu\left(\Sigma Nsin\frac{2Np\nu}{T_1}\right)\left(\Sigma Nsin\frac{2Np\nu}{CT_1}\right)\left[Cos\frac{2\pi}{T_1}\left\{(1+\frac{1}{C})t+(\gamma_1+\frac{\gamma_1'}{C})\right\}\right.$$

$$\left.+cos\frac{2\pi}{T_1}\left\{(1-\frac{1}{C})t+(\gamma_1-\frac{\gamma_1'}{C})\right\}\right]$$

$$-k_1p\nu\left(\frac{Sin\frac{(2n+1)\pi p\nu}{T_1}}{Sin\frac{\pi p\nu}{T_1}}\right)\left(\Sigma NSin\frac{2Np\nu}{CT_1}\right)\left[Sin\frac{2\pi}{T_1}\left\{(1+\frac{1}{C})t+(\gamma_1+\frac{\gamma_1'}{C})\right\}\right.$$

$$\left.+Sin\frac{2\pi}{T_1}\left\{(1-\frac{1}{C})t+(\gamma_1-\frac{\gamma_1'}{C})\right\}\right]-k_1'p\nu\left(\frac{Sin\frac{(2n+1)\pi p\nu}{T_1}}{Sin\frac{\pi p\nu}{T_1}}\right)\left(\Sigma NSin\frac{2Np\nu}{T_1}\right)\times$$

$$\left[Sin\frac{2\pi}{T_1}\left\{(1+\frac{1}{C})t+(\gamma_1+\frac{\gamma_1'}{C})\right\}-Sin\frac{2\pi}{T_1}\left\{(1-\frac{1}{C})t+(\gamma_1-\frac{\gamma_1'}{C})\right\}\right]\right\} \quad (21)$$

$$\therefore \frac{1}{p\nu}\int_0^{p\nu} M_t^2\,dt=\frac{a_1^2 e^{-2K_1 np\nu}}{2(2n+1)^2}\left\{\left(\frac{Sin\frac{(2n+1)\pi p\nu}{T_1}}{Sin\frac{\pi p\nu}{T_1}}\right)^2+4k_1^2p^2\nu^2\left(\Sigma NSin\frac{2N\pi p\nu}{T_1}\right)^2\right\}$$

$$+\frac{a_1'^2 e^{-2K_1'np\nu}}{2(2n+1)^2}\left\{\left(\frac{Sin\frac{(2n+1)\pi p\nu}{CT_1}}{Sin\frac{\pi p\nu}{CT_1}}\right)^2+4k_1'^2p^2\nu^2\left(\Sigma NSin\frac{2N\pi p\nu}{CT_1}\right)^2\right\}$$

$$+\frac{\sigma_{ai}^2}{2n+1} \quad (22)$$

And the standard deviation of u's is

$$\sigma^2=\frac{1}{(2n+1)p\nu}\int_0^{(2n+1)p\nu}\Big[a_1^2e^{-2K_1t}Sin^2\frac{2\pi}{T_1}(t+\gamma_1)+a'^2_1e^{-2K'_1t}sin^2\frac{2\pi}{CT_1}(t+\gamma'_1)$$

$$+a_1a_1'e^{-K(_1+K'_1)t}\left(sin\frac{2\pi}{T_1}\Big\{(1+\frac{1}{C})t+(\gamma_1+\frac{\gamma_1'}{C})\Big\}-cos\frac{2\pi}{T_1}\Big\{(1-\frac{1}{C})t+(\gamma_1+\frac{\gamma_1'}{C})\Big\}\right)\Big]dt$$

$$\doteqdot a_1^2e^{-2K_1np\nu}+a'^2_1e^{-2K_1'np\nu}+\sigma_{ai}^2 \qquad (23)$$

So that the periooagrm equation is given by–

$$\eta^2=\frac{\frac{a_1^2}{2}e^{-2K_1np\nu}\left\{\left(\frac{\frac{Sin(2n+1)\pi p\nu}{CT_1}}{Sin\frac{\pi p\nu}{CT_1}}\right)^2+4k'^2_1p^2\nu^2\left(\Sigma Nsin\frac{2N\pi p\nu}{CT_1}\right)^2\right\}}{\frac{a_1^2}{2}e^{-2K_1np\nu}+\frac{a_1'^2}{2}e^{-2K_1'np\nu}+\sigma_{ai}^2}$$

$$+\frac{a'^2_1e^{-2K_1'np\nu}\left\{\left(\frac{\frac{Sin(2n+1)\pi p\nu}{CT_1}}{Sin\frac{sin}{CT_1}}\right)^2+4K'^2_1p^2\nu^2\left(\sum NSin\frac{2N\pi p\nu}{CT_1}\right)^2\right\}}{\frac{a_1^2}{2}e^{-2K'_1np\nu}+\frac{a'^2_1}{2}e^{-2K'_1np\nu}+\sigma_{ai}^2} \qquad (24)$$

The Maximum value of η^2 is given as the next–

$$\eta^2_{max}=\frac{\sum^2_{mCT_1}}{\sigma^2_{CT_1}}=\frac{\frac{a^2_1}{2}e^{-2K_1np\nu}+\frac{a'^2_1}{2}e^{-2K_1'np\nu}+\frac{\sigma^2_{ai}}{2n+1}}{\frac{a^2_1}{2}e^{-2K_1np\nu}+\frac{a'^2_1}{2}e^{-2K_1'np\nu}+\sigma_{ai}^2}\quad for\ p\nu=CT_1 \qquad (25)$$

where $\begin{cases}\Sigma_{mCT_1} = \text{standard deviation of } M's\ for\ p\nu=CT_1\\ \sigma_{CT_1} = \text{standard deviation of } u's\ for\ p\nu=CT_1\end{cases}$

Now we adopt the central term which is used in calculation of $\Sigma^2_{mCT_1}$ and $\sigma^2_{CT_1}$ of (25) as the central term, and we try the same method as the expression (16).

Then we have

$$a_1^2e^{-2K'nCT_1}\equiv A$$

So that

$$a_1'e^{-K_1'nCT_1}\doteqdot\sqrt{\frac{2n+1}{n}\Sigma^2_{mCT_1}-\frac{\sigma^2_{CT_1}}{n}-A} \qquad (26)$$

But a'_1 *is no existence for*

$$\frac{2n+1}{n}\Sigma^2_{mCT_1}-\frac{\sigma^2_{CT_1}}{n}-A\leqq 0$$

Next we try the above method bynsing u's from the second line to the $(2n+2)$ th line, and we have

$$a'_1 e^{-K_1'(n+1)CT_1}=\sqrt{\frac{2n+1n}{n}\Sigma'^2_{mCT_1}-\frac{\sigma'^2_{mCT_1}}{n}-A'} \tag{27}$$

where
Σ_{mCT_1} = standard deviation of M'_s *for* $p\nu=CT_1$,
σ_{CT_1} = standard deviation of u'_s *for* $p\nu=CT_1$,
$A^1=e_1^2 e^{-2K_1(n+1)CT_1}$

From (26) and (27), logarithmic decrement is given as follows,-

$$-\frac{CT_1K'_1}{2}=\frac{1}{4}\left\{log\left(\frac{2n+1}{n}\Sigma'^2_{mCT_1}-\frac{\sigma'^2_{CT_1}}{n}-A'\right)-log\left(\frac{2n+1}{n}\Sigma^2_{mCT_1}-\frac{\sigma^2_{CT_1}}{n}-A\right)\right\} \tag{28}$$

Again amplitude a'_1 is given as follows,-

$$a'_1=\sqrt{\frac{2n+1}{n}\Sigma^2_{mCT_1}-\frac{\sigma^2_{CT_1}}{n}-A}$$
$$\times\left[1+\frac{n}{2}\left\{log\left(\frac{2n+1}{n}\Sigma^2_{mCT_1}-\frac{\sigma^2_{CT_1}}{n}-A\right)-log\left(\frac{2n+1}{n}\Sigma'^2_{mCT_1}-\frac{\sigma'^2_{CT_1}}{n}-A'\right)\right\}\right] \tag{29}$$

When there is a component wave with a period bT_1 which has an integral multiple period of CT_1, we can get the result as the same method as the method of (26), (27), (28) and(29).

Conclusion

From the above reasoning we can conclude shortly:-

(1) To find the periods of component waves is by drawing the graph of Periodgram Equation. Of course this method is notnecessary, but paying

attention to the maximum values of η^2 is sufficient.

(2) Existence of component wave is decided by calculation.

(3) Logarithmic decrements and amplitudes of component waves are had by calculation.

Application

Material:-

An occilogram of a bell 汰 which is oneof Korean Shrine Festival musical instruments at Confucius Shrine in Seoul, and we used the properly enlarged parts of the occilogram which is taken by Electro-magnetic Occilograph.

Example (A)

After striking, $\left(\frac{0.3}{100} \sim \frac{4.3}{100}\right)$ sec.

Enlarged rate, time interval $\frac{1}{100}$ sec. to 93 cm.

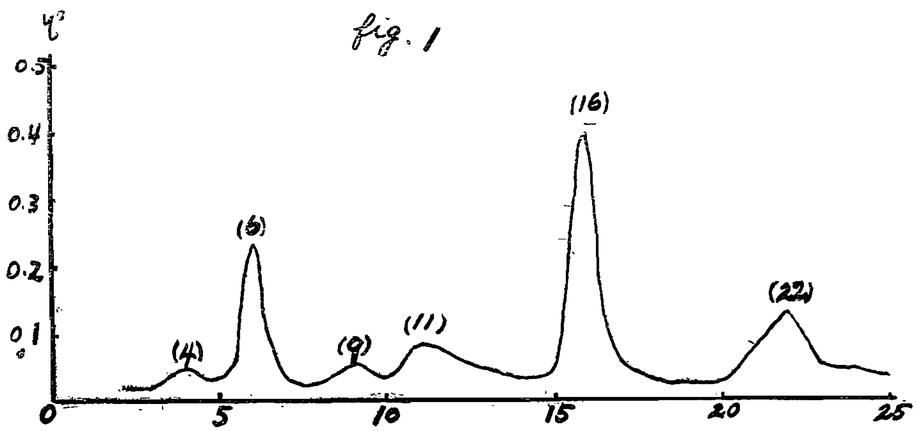
We get table (I) and (II) from the above material.

Table (I) sound wave material of Bell 汰

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	43	46.0	50	65	45	27	29	60	57	45	40	36.5	45	54	55	55	54	40	40	60	75.0	45.0	32.5	32	40
2	60	55.0	45	35	30	42	70	77	50	23	37	70.0	65	55	42	28	43	55	47	56	45.0	20.0	25.0	70	77
3	62	25.0	30	45	72	67	40	35	46	41	48	65.0	56	30	32	55	67	73	44	26	41.0	61.0	64.0	55	40
4	33	34.0	64	56	42	30	60	70	65	46	33	48.0	59	56	43	35	34	34	70	45	22.0	30.0	55.0	75	50
5	26	30.0	61	56	49	26	26	45	66	60	45	23.0	50	80	64	45	34	35	55	70	61.0	27.0	18.5	40	72
6	70	33.0	30	60	64	62	62	51	26	40	70	60.0	16	28	60	79	40	22	30	62	70.0	60.0	40.0	33	33
7	46	73.0	60	22	30	50	69	59	35	30	40	59.0	71	50	27	31	55	70	55	20	27.0	60.0	63.0	50	39
8	40	39.5	53	66	56	23	22	55	85	80	10	25.0	49	66	65	30	35	47	50	62	75.0	45.0	17.0	71	72
9	40	18.0	28	65	74	55	35	30	45	57	93	50.0	13	35	75	67	40	20	35	56	62.5	66.5	51.0	25	15
10	65	91.0	60	18	6	75	85	45	25	32	45	60.0	65	54	35	25	40	85	67	15	12.0	30.0	70.0	69	34
11	24	31.0	48	63	70	20	20	50	79	83	65	49.0	20	40	62	59	44	43	31	22	75.0	88.0	48.0	22	20
12	90	70.0	10	20	53	64	58	30	26	36	55	70.0	70	25	10	48	95	90	23	10	30.0	50.0	82.0	43	29
13	25	45.0	75	87	25	20	47	63	82	65	20	22.0	39	65	67	55	5	10	40	90	66.0	54.0	10.5	50	81
14	80	45.0	31	25	40	60	69	61	35	10	80	91.0	60	16	19	65	82	60	30	30	39.0	58.0	70.0	30	18
15	35	72.0	78	61	25	35	63	62	56	35	11	20.0	75	80	45	18	19	75	84	35	29.0	50.0	56.0	63	60
16	51	40.0	20	45	78	75	20	10	60	88	65	36.0	35	34	55	74	50	22	20	59	80.0	24.0	21.0	36	77

Table (Ⅱ) Table of $\eta^2 = \frac{\Sigma^2}{\sigma^2}$

	2	3	4	5	6	7	3	9	10	11	12	13	14
μ^2	0.005	0.005	0.039	0.015	0.219	0.009	0.014	0.041	0.023	0.083	0.064	0.046	0.021
	15	16	17	18	19	20	21	22	23	24	25		
	0.038	0.395	0.038	0.029	0.023	0.027	0.082	0.128	0.044	0.042	0.024		



From the graph (fig. 1) by the second table, we have six researching needful points of (4), (6), (9), (11), (16) and (22)

At (4) n=7, $\frac{15}{7}\Sigma^2 - \frac{\sigma^2}{7} = -12.10 < 0$

$\therefore$ a_4 is no existence.

At (6) $\frac{15}{7}\Sigma^2 - \frac{\sigma^2}{7} = 62.41$

$$\frac{15}{7}\Sigma'^2 - \frac{\sigma'^2}{7} = 47.36$$

$$-\frac{TK}{2} = -0.069$$

$$a_6 = 15.52$$

frequency = 1550/sec.

At (9) $\frac{15}{7}\Sigma^2-\frac{\sigma^2}{7}=-11.84<0$

a_9 is no existence.

At (11) $\frac{15}{7}\Sigma^2-\frac{\sigma^2}{7}=8.60$

$\frac{15}{7}\Sigma'^2-\frac{\sigma'^2}{7}=1.70$

$-\frac{TK}{2}=-0.176$

$a_{11}=19.54$

frequency$=846$/sec.

the amplitude decreases repidly.

At (22) $\frac{15}{7}\Sigma^2-\frac{\sigma^2}{7}=49.31$

$A\equiv\frac{31}{15}\Sigma_1^2-\frac{\sigma_1^2}{15}=0.02$

$\frac{15}{7}\Sigma'^2-\frac{\sigma'^2}{7}=57.38$

$A'\equiv\frac{31}{15}\Sigma_1'^2-\frac{\sigma_1'^2}{15}=0.01$

So $\frac{15}{7}\Sigma^2-\frac{\sigma^2}{7}-A=49.29$

$\frac{15}{7}\Sigma'^2-\frac{\sigma'^2}{7}-A'=57.37$

$-\frac{TK}{2}=0.0378$

frequency$=423$/sec.

$a_{22}=2.59$

$$\therefore\quad \frac{a_6}{79}=\frac{a_{11}}{100}=\frac{a_{16}}{25}=\frac{a_{22}}{18}$$

Example (B)

After striking, $\left(\frac{19}{100}\sim\frac{23.1}{100}\right)$sec.

Enlarged rate, time interval $\frac{1}{100}$ sec. to 99.5cm.

We have table (III) and (IV) from the above material.

Table (III) sound wave material of Bell 汰

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	29.0	31.0	54	61	54.0	36.0	35.0	43	54	61.0	49	31	31.0	40.0	43.5	48	49.0	37.0	26	32.0	45	54.0	60	41	31.0
2	36.0	55.0	64	49	33.0	29.5	32.5	37	51	58.0	37	26	28.0	36.0	65.0	66	50.0	26.0	32	42.0	60	59.0	40	29	25.5
3	29.0	56.0	61	54	32.0	24.0	45.0	64	70	55.0	28	30	40.0	56.0	58.0	50	30.0	23.5	25	55.0	60	51.0	31	20	40.0
4	67.0	65.0	43	36	35.5	40.0	57.0	52	40	32.0	29	31	55.0	58.0	43.0	30	31.0	50.0	58	50.0	43	36.0	39	47	47.0
5	42.5	37.0	26	25	45.0	58.0	45.0	35	33	45.0	60	52	39.0	42.0	41.0	42	46.5	48.0	40	28.5	23	40.0	50	61	45.0
6	32.0	34.0	60	65	45.0	38.0	37.0	38	46	55.0	51	38	26.0	26.0	54.0	66	45.0	32.0	27	37.0	55	63.0	56	45	35.0
7	29.0	35.0	68	60	43.0	21.0	20.0	30	59	68.0	60	35	27.0	40.0	56.0	59	55.0	45.0	34	25.0	56	61.0	57	40	19.0
8	28.0	67.0	60	44	36.5	36.5	43.0	57	59	50.0	37	31	35.0	58.0	60.0	40	30.0	26.0	50	56.0	49	43.5	36	38	55.0
9	54.5	49.5	46	33	31.0	50.0	55.0	35	32	29.0	45	55	45.5	42.0	40.0	41	46.0	54.0	53	45.0	37	28.0	50	59	43.0
10	31.0	26.0	50	61	54.0	40.0	37.0	40	46	57.0	54	51	32.0	26.0	35.0	53	61.0	45.0	30	21.0	31	60.0	58	46	36.0
11	31.0	40.0	60	67	59.0	43.0	26.0	30	57	62.0	54	34	22.0	42.0	57.0	61	49.0	39.0	29	41.0	65	65.0	49	30	27.0
12	50.0	54.0	55	34	29.5	34.0	52.0	59	57	44.0	33	44	52.0	64.0	51.0	28	30.0	45.0	52	47.0	39	31.0	35	50	59.0
13	53.0	42.0	38	40	50.0	58.0	50.0	37	33	40.6	45	49	49.0	37.0	32.5	45	55.0	58.0	50	35.0	36	53.0	60	51	36.0
14	30.5	36.0	50	53	45.0	34.0	30.0	45	56	61.5	55	39	30.0	53.0	62.0	55	42.0	27.0	26	46.0	57	50.5	36	27	37.0
15	31.0	38.0	53	30	31.0	45.5	54.0	58	51	31.0	28	43	52.0	61.5	48.0	23	35.0	55.0	66	61.5	45	34.0	39	48	59.0
16	50.0	31.0	27	37	52.0	62.0	60.0	28	34	47.0	58	51	32.5	39.0	46.5	53	53.0	40.0	31	31.0	45	55.0	56	40	40.0

Table (IV) Table of $\eta^2 = \frac{\Sigma^2}{\sigma^2}$,

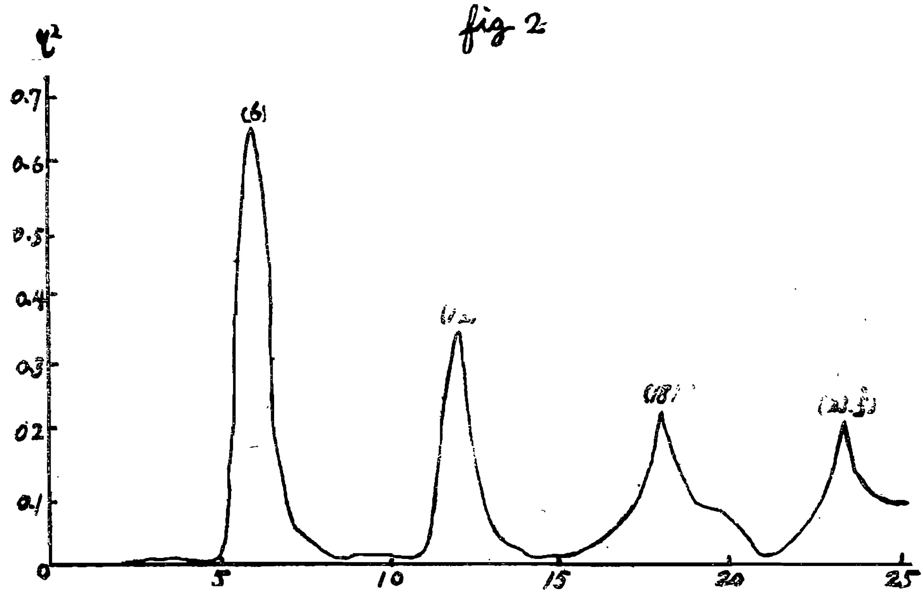
	2	3	4	5	6	7	8	9	10	11	12	13	14
η^2	0.002	0.014	0.004	0.016	0.650	0.069	0.020	0.019	0.018	0.014	0.347	0.047	0.015

15	16	17	18	19	20	21	22	23	23.5	24	25
0.011	0.030	0.064	0.220	0.084	0.076	0.009	0.033	0.113	0.210	0.118	0.099

From the graph (Fig. 2) by the fourth table, we have four researching needful pints of (6), (12), (18) and (23.5).

At (6) $\quad \frac{15}{7}\Sigma^2 - \frac{\sigma^2}{7} = 240.34$

fig 2



$$\frac{15}{7}\Sigma'^2-\frac{\sigma'^2}{7}=247.30$$

$$-\frac{KT}{2}=0.00718$$

$$a_6=15.11$$

frequency=1658/sec.

At (12)

$$\frac{15}{7}\Sigma^2-\frac{\sigma^2}{7}=96.32$$

$$\frac{15}{7}\Sigma'^2-\frac{\sigma'^2}{7}=77.78$$

$$A\equiv a_6e^{-K_1nCT_1}\equiv a_6e^{-2K_1nT_1}=\frac{31}{15}\Sigma_1^2-\frac{\sigma_1^2}{15}=171.36$$

$$A'\equiv a_6e^{-K_1(n+1)CT_1}\equiv a_6e^{-2K_1(n+1)T_1}=\frac{31}{15}\Sigma_1'^2-\frac{\sigma_1'^2}{15}=165.48$$

then $$\frac{15}{7}\Sigma^2-\frac{\sigma^2}{7}-A=-75.04<0$$

So that a_{12} is no existence. But by the research for the changing state of a component whose amplitude a_6, We rave

$$-\frac{KT}{2}=-0.0086$$

$$a_6=15.77$$

It shows, the component wave which has an amplitude a_6 is in decreasing state through increasing state. and it has a nearly equal amplitude in the time interval of 15 times of its own period.

At (18)

$$\frac{15}{7}\Sigma^2-\frac{\sigma^2}{7}=49.14$$

$$\frac{15}{7}\Sigma'^2-\frac{\sigma'^2}{7}=47.77$$

$$A\equiv\frac{47}{23}\Sigma_2^2-\frac{\sigma_2^2}{23}=36.70$$

$$A'\equiv\frac{47}{23}\Sigma'^2_2-\frac{\sigma'^2_2}{23}=36.58$$

$$\therefore\ \frac{15}{7}\Sigma^2-\frac{\sigma^2}{7}-A=12.44$$

$$\frac{15}{7}\Sigma'^2-\frac{\sigma'^2}{7}-A'=11.19$$

So we have

$$-\frac{KT}{2}=-0.02932$$

$$a_{18}=4.71$$

freqnency=553/sec.

At (23.5)

$$\frac{15}{7}\Sigma^2-\frac{\sigma^2}{7}=1.54$$

$$\frac{15}{7}\Sigma'^2-\frac{\sigma'^2}{7}=1.79$$

$$-\frac{KT}{2}=0.0375$$

$$a_{23.5}=0.588$$

$$\text{frequency}=424/\text{sec},$$

and so

$$\frac{a_6}{100}=\frac{a_{18}}{31}=\frac{a_{23.5}}{4}$$

Example (C)

After striking $\left(\frac{39}{100}\sim\frac{43}{100}\right)$ sec.

Enlarged rate, time interval $\frac{1}{100}$ sec. to 99. 5 cm.

We have the table (V) and the (VI) from the above material.

Table (V) sound wave material of Bell 汰

	1	2	3	4	5	6	7	8	9	10	11	12	13
1	33.0	40.0	47.0	55.0	50.0	42.0	42.5	46.5	53.0	53.5	41.0	36.0	38.5
2	51.0	52.0	47.0	39.0	37.0	38.5	43.5	45.5	45.5	42.0	35.0	38.0	47.5
3	49.0	49.0	39.0	33.0	37.5	45.0	53.0	57.0	40.0	33.5	40.0	50.0	53.5
4	57.5	44.0	38.0	37.5	42.5	50.0	55.5	40.0	32.0	34.0	40.0	47.5	53.5
5	40.0	35.0	34.0	42.0	50.0	49.0	40.0	39.0	41.0	45.0	51.0	51.0	43.0
6	40.0	42.5	48.0	50.5	44.0	39.5	39.0	43.5	48.5	49.0	40.0	33.0	35.0
7	55.0	49.0	32.0	34.0	41.0	45.0	52.0	52.0	45.0	33.0	37.0	47.0	53.0
8	53.0	39.0	35.5	36.5	50.0	59.0	59.0	40.0	34.0	37.0	47.0	53.0	52.0
9	41.0	38.5	39.0	48.0	52.0	44.0	39.0	36.5	38.0	43.0	49.0	46.0	39.0
10	39.0	40.0	45.0	48.0	44.0	40.0	42.0	48.5	51.5	49.0	42.0	48.0	43.0
11	55.0	56.0	36.0	35.5	40.0	45.5	48.0	45.0	36.0	32.5	36.0	46.0	50.5
12	50.0	40.0	32.0	34.0	40.0	50.0	51.0	40.0	35.0	38.5	45.0	55.5	50.0
13	47.0	40.0	40.5	43.0	52.0	45.0	38.0	37.0	38.5	41.5	46.0	47.0	41.0
14	41.0	40.0	40.2	43.0	47.5	45.0	39.0	38.0	43.0	50.0	50.0	48.0	45.0
15	40.0	41.5	45.5	53.0	51.0	41.0	39.5	44.0	50.0	50.5	49.5	42.0	35.0
16	39.0	41.0	49.0	53.0	50.0	38.0	34.0	40.0	47.5	52.0	49.0	41.0	34.0

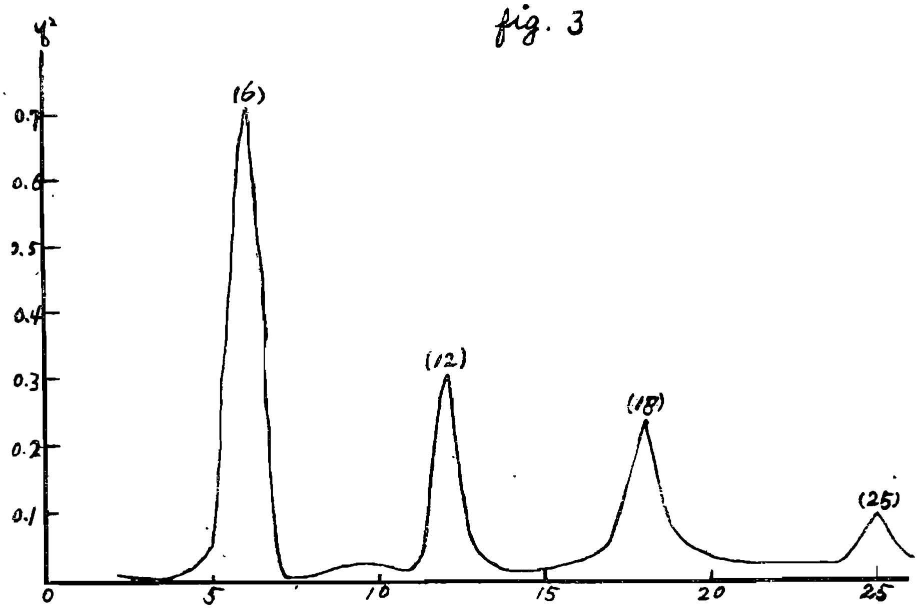
	14	15	17	16	18	19	20	21	22	23	24	25	26
	42.0	44.0	44.0	41.2	37	37.5	44.0	50.5	49.0	44.0	40.0	41.0	
	53.0	52.0	42.0	37.0	40	50.0	55.0	47.0	38.0	35.0	37.0	44.0	
	52.0	40.0	29.0	34.0	48	53.0	48.0	37.0	32.5	35.0	45.0	57.0	
	44.0	33.0	37.0	45.0	54	54.0	46.0	38.0	37.0	46.0	50.0	49.0	
	38.0	39.5	45.0	48.2	45	38.0	35.5	38.0	45.0	48.5	44.0	40.0	
	48.0	50.0	43.0	39.5	37	38.0	45.0	55.0	42.0	37.0	39.0	47.0	

52.0	39.0	33.0	36.0	47	53.5	41.0	35.0	32.5	35.0	42.0	53.0
37.0	32.0	36.0	43.0	50	49.0	43.0	38.0	36.5	52.0	56.0	52.0
37.5	44.0	52.0	52.0	46	42.0	40.0	45.0	47.5	44.0	39.5	40.0
50.0	47.5	41.0	40.0	37	39.0	45.0	50.0	45.0	40.0	41.0	47.0
50.5	45.5	36.0	36.0	52	58.0	51.0	42.0	37.0	36.0	45.0	50.0
40.0	35.0	37.0	45.0	50	48.0	40.0	34.0	33.5	45.0	53.0	55.0
35.5	37.0	46.0	51.5	49	45.0	42.2	43.5	47.0	50.5	48.5	44.0
41.5	46.0	51.5	48.0	44	43.0	40.0	39.0	45.0	50.0	45.0	40.5
34.0	40.0	49.0	43.0	46	37.0	35.0	45.0	54.0	56.0	47.0	41.0
35.0	45.0	53.5	64.0	47	39.0	38.0	40.5	48.0	53.0	49.0	39.0

Table (VI) Table of $\eta^2 = \frac{\Sigma^2}{\sigma^2}$

	2	3	4	5	6	7	8	9	10	11	12	13
η^2	0.005	0.003	0.002	0.062	0.701	0.007	0.009	0.020	0.031	0.015	0.309	0.026
14	15	16	17	18	19	20	21	22	23	24	25	26
0.012	0.019	0.024	0.055	0.233	0.047	0.038	0.022	0.020	0.012	0.017	0.091	0.023

fig. 3



From the graph (fig. 3) which is described by the the table (VI), we have four researching needful pints (6), (12), (18) and (25), -

At (6)

$$\frac{15}{7}\Sigma^2-\frac{\sigma^2}{7}=66.09$$

$$\frac{15}{7}\Sigma'^2-\frac{\sigma'^2}{7}=74.36$$

$$-\frac{KT}{2}=0.02944$$

$$a_6=7.293$$

frequency=1658/sec.

The amplitude is in increasing state.

At (12)

$$\frac{15}{7}\Sigma^2-\frac{\sigma^2}{7}=27.79$$

$$\frac{15}{7}\Sigma'^2-\frac{\sigma'^2}{7}=25.40$$

$$A=24.81$$

$$A'=22.38$$

$$\therefore\quad \frac{15}{7}\Sigma^2-\frac{\sigma^2}{7}-A=2.98$$

$$\frac{15}{7}\Sigma'^2-\frac{\sigma'^2}{7}-A'=3.02$$

$$-\frac{KT}{2}=0.0038$$

$$a_{12}=1.65$$

frequency=829/sec.

At (18)

$$\frac{15}{7}\Sigma^2-\frac{\sigma^2}{7}=3.44$$

$$\frac{15}{7}\Sigma'^2-\frac{\sigma'^2}{7}=3.14$$

$$A=\frac{47}{23}\Sigma_2^2-\frac{\sigma_2^2}{23}=3.00$$

$$A' = \frac{47}{23} \Sigma'^2_2 - \frac{\sigma'^2_2}{23} = 2.77$$

then

$$\frac{15}{7} \Sigma^2 - \frac{\sigma^2}{7} - A = 0.44$$

$$\frac{15}{7} \Sigma'^2 - \frac{\sigma'^2}{7} - A' = 0.37$$

$$-\frac{KT}{2} = -0.043$$

$$a_{18} = 1.06$$

frequency=553/sec.

At (25)

$$\frac{15}{7} \Sigma^2 - \frac{\sigma^2}{7} = 1.94$$

$$\frac{15}{7} \Sigma'^2 - \frac{\sigma'^2}{7} = 2.13$$

$$-\frac{KT}{2} = 0.0231$$

$$a_{25} = 0.939$$

frequency=398/sec.

So we have

$$\frac{a_6}{100} = \frac{a_{12}}{23} = \frac{a_{18}}{14} = \frac{a_{25}}{13}$$

The results of three example shown by acousttic spectrums of fig 4, and a used occilogam of the bell of Korean Shrine Festival musical Instruments at Confucius Shrin in Seoul is shown by fig. 5.

fig. 4
Acoustic Spectrum of Bell 汰

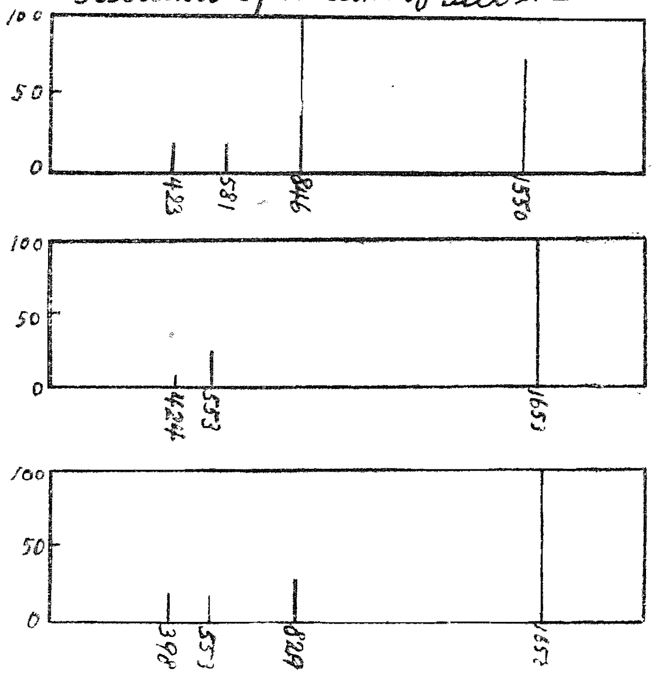
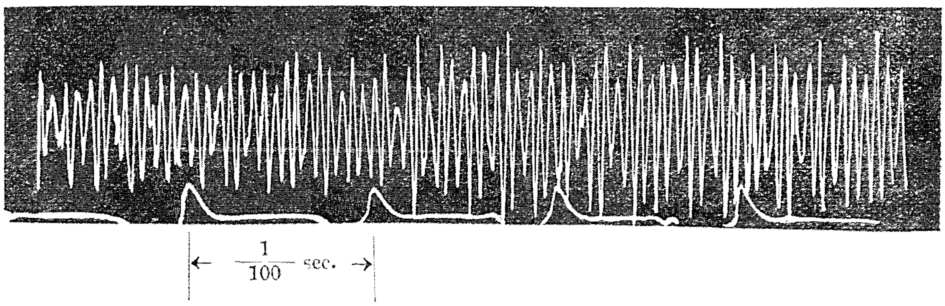
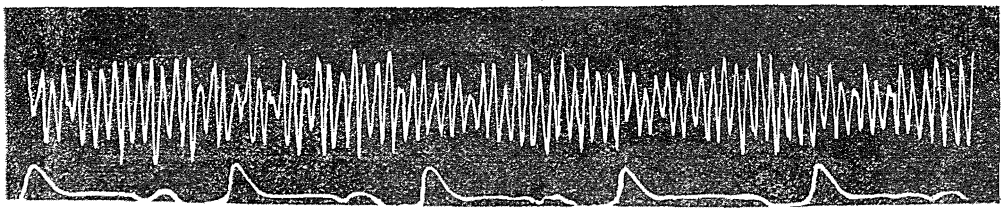


fig 5. Occilogram of a Sound of Bell 汰

A part



B part



C part