ON DIFFERENTIABLE FIBRE BUNDLES WITH A CERTAIN FIELD

TAKUYA SAEKI

或場をもった微分可能なファイバーバンドルについて

佐　伯　卓　也

Recently Y. Mutô [4] [1] and S. Ishihara [3] have studied some theorems concerning fibred Riemannian spaces. They have especially dealt with the I. P. F. Riemannian space which is one of the various types of fibred Riemannian spaces introduced by Y. Mutô. We ahall attempt to generalize some properties of them to differentiable fibre bundles with the affine connection.

For this purpose, we shall first recall the notion of differentiable fibre bundles and introduce a system of favourable coordinates and define a field F of n-dimensional plane-elements by some system of Pfaffian differential equations. Next we shall prove that if the field F is completely integrable (if the bunlle B (X, Y) admits holonomic fibres), then it admits a completely reducible affine connection.

Let us suppose that indices run as follows:

$$a, b, c, \cdots = 1, 2, \cdots, n;$$
$$i, j, k, \cdots = n+1, n+2, \cdots, n+m;$$
$$\alpha, \beta, \gamma, \cdots = 1, 2, \cdots, n+m.$$

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§ 1. Differentiable fibre bundles with a field F with lifts. First of all, we shall recall some properties of differentiable fibre bundles. Let $B(X, Y)$ be a differentiable fibre bundle of class C^∞. In $B(X, Y)$ the structure group G is a group of differentiable transformations of the fibre Y onto itself. We shall consider in the bundle space B a system of special coordinate neighbourhoods, i. e., a system of favourable coordinate neighbourhoods introduced by Y. Mutô [4] .

Let $\{U_\lambda\}$ (resp. $\{V_\tau\}$) be a system of coordinate neighbourhoods of the base space X (resp. fibre Y) and (x^a_λ) (resp. (x^i_τ)) a coordinate system in U_λ (resp. V_τ). We shall define an open set in the bundle space B by $W_{\lambda\tau} = \phi_\lambda (U_\lambda \times V_\tau)$, where ϕ_λ is a coordinate function : $U_\lambda \times Y \to p^{-1}(U_\lambda)$ (p is the projection of B to X). The collection $\{W_{\lambda\tau}\}$ of open sets in B is an open covering of B. If $b \in W_{\lambda\tau}$, then $p(b) = x \in U_\lambda$ and $p_\lambda(b) = y \in V_\tau$ where p_λ is the mapping of $p^{-1}(U_\lambda)$ to Y. Let (x^a_λ) and (x^i_τ) be coordinate systems of $p(b)$ and $p_\lambda(b)$ respectively, then that of b in $W_{\lambda\tau}$ is given by $(x^a_\lambda(b),\ x^i_\tau(p(b)))$ and it easy to see that $(x^a_\lambda(p(b)),\ x^i_\tau(p_\lambda(b)))$ is a coordinate system of class C^∞ in $W_{\lambda\tau}$. We shall denote it by (x^a_λ, x^i_τ) or

[1] Numbers in brackets refer to the bibliography at the end of the paper.

simply (x^a, x^i) and call it by a *favourable coordinate system* of $W_{\lambda\tau}$. We can always cover any differentiable fibre bundle by a set of favourable coordinate systems.

Let (x^a, x^i) and $(x^{a'}, x^{i'})$ be two favourable coordinate systems at a point of B. Then there exists a transformation between these coordinate systems which can be expressed by the equations

$$(1)\qquad \begin{aligned} x^{a'} &= x^{a'}(z^1, x^2, \cdots, x^n), \\ x^{i'} &= x^{i'}(x^1, x^2, \cdots, x^n ; x^{n+1}, x^{n+2}, \cdots, x^{n+m}). \end{aligned}$$

The first set of equations (1) gives a tranformation of coordinates in the base space X, and the second one for fixed (x^a) is nothing but a local expression of a coordinate transformation of the fibre bundle $B(X, Y)$, i. e.,

$$y' = g_{\mu\lambda}(x)\, y$$

where $y, y' \in Y$, $x \subset U_\lambda \cap U_\mu \subset X$, U_λ, U_μ are two intersecting coordinate neighbourhoods and $g_{\mu\lambda} \in G$.

Let x and $x+dx$ be two consecutive points of a coordinate neighbourhood in X and let Y_x and Y_{x+dx} be fibres over x and $x+dx$ respectively. When a point of Y_x corresponds to a point $b+db$ of Y_{x+dx} in the same favourable coordinate neighbourhood and coordinates of b and $b+db$ are (x^a, x^i) and (x^a+dx^a, x^i+dx^i) respectively, we shall define the relation between dx^i' s and dx^a' s by the equations

$$(2)\qquad dx^i = -\Gamma^i_a\, dx^a$$

where Γ^i_a' s are differentiable functions of class C^∞ of x^a and x^i. It is well known that the law of transformation of the quantity Γ^i_a is given by

$$(3)\qquad \Gamma^i_a = \Gamma^{i'}_{a'} \frac{\partial x^{a'}}{\partial x^a} \frac{\partial x^i}{\partial x^{i'}} + \frac{\partial x^i}{\partial x^{j'}} \frac{\partial x^{j'}}{\partial x^a}.$$

The system of Pfaffian differential equations (2) define a field F of n-dimensional plane–elements in B. If the manifold admits a Riemannian metric $g_{\alpha\beta}$ with the asumption det $|g_{ij}| \neq 0$, there exists a field $\bar{F}$ of n–dlmensional plane–elements which are orthogonal to the tangent space of the fibre at each point of B. The field $\bar{F}$ is an example of the field F. In fact, $\bar{F}$ is defined by the equations

$$g_{ij}\, dx^j + g_{ia}\, dx^a = 0.$$

Since det $|g_{ij}| \neq 0$ we can solve this and get

$$(4)\qquad dx^i + \bar{\Gamma}^i_a\, dx^a = 0,$$

where $\bar{\Gamma}^i_a$' s are defined by $\bar{\Gamma}^i_a\, g_{ij} = g_{ja}$. The equations (3) correspond to (2). We shall give another example of the field F in § 1 of [7].

Now let us consider the integral curve of the field F. Let C be an arbitrary differentiable curve joining x to x', where $x, x' \in X$. When there exists an integral curve which covers C of the field F joining $b \in Y_x$ to $b' \in Y_{x'}$, where b is an arbitrary point of Y, it is called that a differentiable curve C has the lift. In the differentiable fibre bundle $B(X, Y)$, if any differentiable curve in the base space X has the lift, $B(X, Y)$ is called a differentiable fibre bundle with a field F with lifts.

Let $B(X, Y)$ be a defferentiable fibre bundle with a field F with lifts. Let C be a piecewise differetiable curve in the base space X joining x_0 to x_1. There exists the lift $\tilde{C}$ which starts $b_0 \in Y_{x_0}$, we shall denote the end point of $\tilde{C}$ by b_1, $b_1 \in Y_{x_1}$. We associate to $b_0 \in Y_{x_0}$ the point $b_1 \in Y_{x_1}$, hence this correspoindence gives a mapping; $Y_{x_0} \to Y_{x_1}$ which depends X_0 and X_1 of the bese space X and on the curve C. We shall denote mapping by $\varphi(C)$, It is easy to see that $\varphi(C)$ is differentiable.

Since the base space X is a manifold, we can always select a system of coordinate neighbourhoods of X such that each of them are simply connected and that the intersection of any two neighbourhoods is connected if they intersect. Hence for a piecwise differentiable curve C, which starts x_0 and ends x_1, we can cover C by suitable fine simply connected neighbourhoods. Let C be a curve which is covered by the above mensioned neighbourhoods, if the mapping $\varphi(C)$ only depends on two end points of the curve C and not on any curves such that in the same neighbourhoods of C joining these two points, then the system of Pfaffian differential equations (2) is completely integrable. The converse is also true. The above statement is the meaning of complete integrability of the field F with lifts.

Let $\{C\}$ be a set of all closed piecewise differentiable curves which pass hrough a fixed point $x_0 \in X$. Then the set $\{\varphi(C)\}$ of mapping mensioned above forms a group.[2)]

If the field F is completly integrable, we can consider the maximal integral manibold. Let Y_x be the fibre on $x \in X$. We shall denote the maximal integral manifold passing through $q_0 \in Y_x$ by $\tilde{X}(b_0)$.

Proposition 1. *When the field F with lifts in $B(X, Y)$ is completely integrable, the maximal integral manifold $\tilde{X}(b_0)$ passing through a point $b_0 \in Y_x$ forms a covering space of X whose covering mapping is $\tilde{p} = p \mid \tilde{X}(b_0)$.*[3)]

Proof. It is easy to see that $\tilde{p} = p \mid \tilde{X}(b_0)$ is a differentiable map of $\tilde{X}(b_0)$ onto X. Now we need only show that each point $x \in X$ has a neighbourhood U such that $\tilde{p}$ defined a differentiable homeomorphism of each component $\tilde{p}^{-1}(U)$ with U.

For $x \in X$, we can select a sufficiently small simply connected neighbourhood U of x. Let x' be a point of U and C' be a piecewise differentiable curve joining x to x'. Let b be a point of $\tilde{X}(b_0)$ such that $\tilde{p}(b) = x$. There exists a curve which covers C' in $\tilde{X}(b_0)$, and let b' be the end point of C'. Clearly $\tilde{p}(b') = x'$ and b' is uniquely determined for $x \in X$ since the field F is completely integrable by assumption. Let $\tilde{U}$ be a set of such b' when x' runs U. Hence $\tilde{U}$ is a component of $\tilde{p}^{-1}(U)$ and $\tilde{p}$ is a homeomorphism of $\tilde{U}$ onto U. q. e. d.

§ 2. **Affine connections.** S. Ishihara has proved the following lemma concerning the I. P. F. (isometric parallel fibred) Riemannian space[4)].

Lemma 1. *An I. P. F. Riemannian space B has a bundle structure having the following properties:*

1^0 *The base space X and the fibre space Y are Riemannian spaces;*

2) C. f. p. 244 of Ishihara [3].
3) C. f. p. 306 of Saeki [6].
4) C. f. Ishihara [3].

2° *The structure group G of the bundle is a Lie group of isometric homeomorphism acting on Y;*

3° *The coordinate transformation $g_{\mu\lambda}$ of the fibre bundle is a constant function on any connected component of $U_\lambda \cap U_\mu$, where U_λ and U_μ are two intersecting coordinate neighbourhoods of X.*

But he remarked that it is sufficient to prove the conditiion 3° only the complete integrability of the field F and we do not need the Riemannian metric.

LEMMA 2. *If the field F with lifts in B (X, Y) is completely integrable, then we can choose a set of favourable coordinate systems in B such that all Γ^i_a vanish identically.* [5]

PROPOSITION 2. *A field F with lifts of a differentiable fibre bundle B (X, Y) is completely integrable if and only if quantities Γ^i_a vanish in all set of favourable coordinate system in B.*

Proof. If $\Gamma^i_a = 0$ in each favourable coordinate neighbourhoods, the field F is comdletely integrable. For, a condition of complete integrability of the system of Pfaffian differential equations (2) is

$$R^i_{ab} \equiv \Gamma^i_{a,b} - \Gamma^i_{a,k}\Gamma^k_b - \Gamma^i_{b,a} + \Gamma^i_{b.k}\Gamma^k_a = 0 \tag{5}$$

and it holds good.

Conversely, if the field F is completely integrable, in the same way as Lemma 1 by the reason we have remarked already, we can show the coordinate transformation $g_{\mu\lambda}$ of the bundle B is a constant function on $U_\lambda \cap U_\mu$, where U_λ and U_μ are two intersecting coordinate neighbourhoods of X.

Then the transformation between two favourable coordinates ystems are expressed by the equations.

$$\begin{aligned} x^{a'} &= x^{a'}(x^1, x^2, \cdots, x^n), \\ x^{i'} &= x^{i'}(x^{n+1}, x^{n+2}, \cdots, x^{n+m}), \end{aligned} \tag{6}$$

where (x^α) and $(x^{\alpha'})$ are two favourable coordinate systems of a point of B. Since functions of the type $\partial x^{i'}/\partial x^a$ are all zero, we can easily show that the quautity Γ^i_a behave like components of a tensor for the transformatiosn (6). By virtue of Lemma 2, we can choose a set of favourable coordinate systms such that all Γ^i_a vanish identically. Hence Γ^i_a vanish for all of favourable coordinate systems in B. q. e. d.

A given affine connection Γ in the manifold B (X, Y) is called to be *completey reducible* if the connection coefficients $\Gamma^\alpha_{\beta\gamma}$ satisfy the following relations:

$$\Gamma^a_{bc}(x^a, x^l) \equiv \Gamma^a_{bc}(x^d), \; \Gamma^i_{jk}(x^d, x^l) \equiv \Gamma^i_{jk}(x^l),$$

the remaining $\Gamma^\alpha_{\beta\gamma}(x^d, x^l)$ are all zero in a set of favourable coordinate neighbourhoods which covers B.

THEOREM. *In a differentiable fibre bundle B (X, Y) with a field F with lifts, if the field F is completely integrable. Then the bundle B (X, Y) admits a completely reducible affine connection Γ.*

Proof. In the same way as the proof of Proposition 1, we have the equations (6) as the transformation between two favourable coordinate systems. Since functions of the type $\partial x^{i'}/\partial x^{a}$ and $\partial x^{a'}/\partial x^{i}$ are all zero, we can easily show that connection coefficients Γ^{i}_{bk}, Γ^{i}_{jc} and Γ^{i}_{bc} of any affine connection on B behave like components of tensors for the coordinate transformations (6), and quantities $\Gamma^{i}_{jk,a}$ and $\Gamma^{a}_{bc,i}$, where $\Gamma^{i}_{jk,a} \equiv \partial \Gamma^{i}_{jk}/\partial x^{a}$ and $\Gamma^{a}_{bc,i} \equiv \partial \Gamma^{a}_{bc}/\partial x^{i}$, also behave like components of tensors for (6).

We shall introduce arbitrary affine connections of X and on Y and denote their connection coefficients by $\Gamma^{a}_{be}\ (x^{d})$ on X and $\Gamma^{i}_{jk}\ (x^{l})$ on Y respectively. An affine connection Γ on B is given by the above $\Gamma^{a}_{bc}\ (x^{d})$, $\Gamma^{i}_{jk}\ (x^{l})$ and the remaining $\Gamma^{\alpha}_{\beta\gamma}$ $(x^{d},\ x^{l}) \equiv 0$. Γ is well-defined, since the connection coefficieots Γ^{i}_{bk}, Γ^{i}_{jc} and Γ^{i}_{bc} and the quantities $\Gamma^{i}_{jk,a}$ and $\Gamma^{a}_{bc,i}$ behave like co mponents of tensors for (4), and Γ is unique by the construction. Hence the manifold $B\ (X, Y)$ admits a completely reducible affine conncetion Γ. q. e. d.

We shall now consider the inverse proposition. Modifying the similar result of Y. Mutô and K. Yano [5] , we can easily show the following lemma.

LEMMA. *If $\Gamma^{a}_{bc}\ (x^{d})$ (resp. $\Gamma^{i}_{jk}\ (x^{l})$) are the connection coefficients on X (resp. Y), then an affine connection Γ' on B are determined by following coefficients:*

$$
\begin{aligned}
&\Gamma'^{i}_{jk} = \Gamma^{i}_{jk} \\
&\Gamma'^{i}_{bk} = \Gamma^{i}_{b,k} + \Gamma^{i}_{jk}\Gamma^{j}_{b}, \quad \Gamma'^{i}_{jc} = \Gamma^{i}_{c,j} + \Gamma^{i}_{jk}\Gamma^{k}_{c} \\
(7)\qquad &\Gamma'^{i}_{bc} = \Gamma^{i}_{b,c} - \Gamma^{a}_{bc}\Gamma^{i}_{a} + \Gamma^{l}_{b}\,(\Gamma^{i}_{c,l} + \Gamma^{i}_{lk}\Gamma^{k}_{c}) \\
&\Gamma'^{a}_{jk} = 0, \quad \Gamma'^{a}_{jc} = 0, \quad \Gamma'^{a}_{bk} = 0, \\
&\Gamma'^{a}_{bc} = \Gamma^{a}_{bc}.
\end{aligned}
$$

PROPOSITION 3. *Let Γ' be an affine connection in a differentiable fibre bundle $B\,(X, Y)$ with a field F with lifts, introduced by the same way as the above lemma from symmetric affine connections on the base space X and on the fibre Y. If the affine connection Γ' is completely reducible in the same coordinate systems as introducing of Γ', then the field F is completely integrable.*

Proof. By the assumption the affine connection Γ' is completely reducible in the same coordinate systems as the introduction of Γ', hence coefficients Γ'^{i}_{bk}, Γ'^{i}_{jc} and Γ'^{i}_{bc} are all zero, hence from the equations (7) we have

$$
\begin{aligned}
(8)\qquad &\Gamma^{i}_{b,k} = -\Gamma^{i}_{jk}\Gamma^{j}_{b}, \quad \Gamma^{i}_{c,j} = -\Gamma^{i}_{jk}\Gamma^{k}_{c} \\
&\Gamma^{i}_{b,c} = \Gamma^{a}_{bc}\Gamma^{i}_{a} - \Gamma^{l}_{b}\,(\Gamma^{i}_{c,l} + \Gamma^{i}_{lk}\Gamma^{k}_{c}).
\end{aligned}
$$

we put (8) into (5) and get

$$
\begin{aligned}
R^{i}_{bc} &= \Gamma^{i}_{b,c} - \Gamma^{i}_{b,k}\Gamma^{k}_{c} - \Gamma^{i}_{c,b} + \Gamma^{i}_{c,k}\Gamma^{k}_{b} \\
&= \Gamma^{i}_{a}\,(\Gamma^{a}_{bc} - \Gamma^{a}_{cb}) + \Gamma^{j}_{l}\Gamma^{k}_{c}\,(\Gamma^{i}_{jk} - \Gamma^{i}_{k,}) - \Gamma^{l}_{b}\Gamma^{c}_{j,l} \\
&\quad - \Gamma^{l}_{b}\Gamma^{k}_{c}\Gamma^{i}_{lk} + \Gamma^{b}_{c}\Gamma^{i}_{b,l} + \Gamma^{k}_{b}\Gamma^{l}_{c}\Gamma^{i}_{lk}
\end{aligned}
$$

But from the equations (8) we have

$$
\Gamma^{l}_{b}\Gamma^{i}_{c,l} = -\Gamma^{l}_{b}\Gamma^{i}_{lk}\Gamma^{k}_{c}, \quad \Gamma^{l}_{c}\Gamma^{i}_{b,l} = -\Gamma^{l}_{c}\Gamma^{i}_{lk}\Gamma^{k}_{b}.
$$

Hence

$$R^i_{bc} = \Gamma^i_a\ (\Gamma^a_{bc} - \Gamma^a_{cb}) + \Gamma^j_b\ \Gamma^k_c\ (\Gamma^i_{jk} - \Gamma^i_{kj})$$

By the assumption the connection coefficients Γ^a_{bc} and Γ^i_{jk} are symmetric, then $R^i_{jk} = 0$ holds good.

We can conclude in every neighbourhood similarly, hence the field F is completely integrable. q. e. d.

If a differentiable fibre bundle $B\ (X, Y)$ with a field F with lifts has a simply connected base space, then it is reducible to a product of two manifolds and then it admits a completely reducible affine connection.

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IWATE UNIVERSITY,
MORIOKA, JAPAN.