DYNAMICAL SYSTEM FOR COMPETITIVE SYSTEM WITH CROSS-DIFFUSIONS

ATSUSHI YAGI

1. Competitive system with cross-diffusion

We consider the initial-boundary value problem for a competitive system

$$
(1) \quad \begin{cases}
\dfrac{\partial u}{\partial t} = \Delta(au + \alpha_{11}u^2 + \alpha_{12}uv) + cu - \gamma_{11}u^2 - \gamma_{12}uv & \text{in } \Omega \times (0, \infty), \\
\dfrac{\partial v}{\partial t} = \Delta(bv + \alpha_{21}uv + \alpha_{22}v^2) + dv - \gamma_{21}uv - \gamma_{22}v^2 & \text{in } \Omega \times (0, \infty), \\
\dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega \times (0, \infty), \\
u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & \text{in } \Omega
\end{cases}
$$

in a two-dimensional bounded $\mathcal{C}^3$ domain $\Omega \subset \mathbb{R}^2$. Here, a, $b > 0$, $\alpha_{ij} \geq 0$ and c, $d > 0$ and $\gamma_{ij} > 0$ are given constants.

As well known, this system has been introduced by Shigesada-Kawasaki-Teramoto [2] to describe the segregation process for two competitive species in a two-dimensional domain Ω by cross-diffusion. The unknown functions $u = u(x,t)$ and $v = v(x,t)$ denote the densities of species in Ω at time $t \geq 0$. The term $\alpha_{ij}\Delta uv$ $(i \neq j)$ denotes the cross-diffusion of each species. On the other hand, the terms $\alpha_{11}\Delta u^2$ and $\alpha_{22}\Delta v^2$ denote the self-diffusions of species. The competitions of species are described by the kinetic functions $(c - \gamma_{11}u - \gamma_{12}v)u$ and $(d - \gamma_{21}u - \gamma_{22}v)v$.

We assume that the cross-diffusion constants α_{12} and α_{21} are smaller than the self-diffusion constants α_{11} and α_{22} in the following specific sense

$$
(2) \qquad 0 \leq \alpha_{12}\alpha_{21} \leq 64\,\alpha_{11}\alpha_{22}.
$$

We may notice however that, when one of the cross-difusion constants vanishes, i.e., $\alpha_{12}\alpha_{21} = 0$, it is allowed that the two self-diffusion constants vanish all together, $\alpha_{11} = \alpha_{22} = 0$.

The space of initial values is given by

$$
(3) \qquad \mathcal{K} = \left\{ \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} ;\ 0 \leq u_0 \in H^{1+\varepsilon}(\Omega) \quad \text{and} \quad 0 \leq v_0 \in H^{1+\varepsilon}(\Omega) \right\},
$$

where ε denotes an arbitrarily fixed exponent in such a way that $0 < \varepsilon < \frac{1}{2}$.

Remark 1. We need the condition (2) only for establishing a priori estimates for local solutions. Therefore we can always construct local solutions without assuming this kind of relations among self and cross diffusion coefficients. □

Remark 2. The $\mathcal{C}^3$ regularity of the boundary $\partial\Omega$ ensures the shift property that $\Delta u \in H^1(\Omega)$ with $\frac{\partial u}{\partial n} = 0$ implies $u \in H^3(\Omega)$. Such a shift property is necessary in the proof of *a priori* estimates in the case when $\alpha_{12}\alpha_{21} = 0$ only. As a matter of fact, when

$0 < \alpha_{12}\alpha_{21} \leq 64\alpha_{11}\alpha_{22}$, the *a priori* estimate can be carried out under the assumption that Ω is a convex domain or its boundary has $\mathcal{C}^2$ regularity. □

2. Local solutions

We have alrady constructed non negative local solution to (1) by applying the theory of abstract nonlinear parabolic evolution equations, see [3, Theorem 3.5]. Indeed, for any $U_0 \in \mathcal{K}$, Problem (1) possesses a unique local solution in the function space

$$(4) \qquad \begin{cases} 0 \leq u \in \mathcal{C}((0, T_{U_0}]; H^2_N(\Omega)) \cap \mathcal{C}([0, T_{U_0}]; H^{1+\varepsilon}(\Omega)) \cap \mathcal{C}^1((0, T_{U_0}]; L_2(\Omega)), \\ 0 \leq v \in \mathcal{C}((0, T_{U_0}]; H^2_N(\Omega)) \cap \mathcal{C}([0, T_{U_0}]; H^{1+\varepsilon}(\Omega)) \cap \mathcal{C}^1((0, T_{U_0}]; L_2(\Omega)), \end{cases}$$

where $T_{U_0} > 0$ is determined by the norm $\|U_0\|_{H^{1+\varepsilon}}$. Here,

$$H^2_N = \{u \in H^2(\Omega);\ \tfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}.$$

Furthermore, by using the maximal regularity of abstract linear parabolic evolution equations, see [3, Theorem 3.6], we can show the regularity

$$(5) \qquad \begin{cases} u,\ v \in \mathcal{C}^1((0, T_{U_0}]; H^{2\theta}(\Omega)), \qquad 0 \leq \theta < 1, \\ u,\ v \in \mathcal{C}^2((0, T_{U_0}]; H^1(\Omega)^*). \end{cases}$$

We conclude this section by announcing the Lipschitz continuity of local solutions with respect to the initial values. This result can also be obtained as a consequence of the general results concerning the abstract parabolic evolution equations, see Theorem 2 of [1]. For $0 < R < \infty$, put

$$\mathcal{K}_R = \{U_0 \in \mathcal{K}; \|U\|_{H^{1+\varepsilon}} < R\}.$$

Then, for any $U_0 \in \mathcal{K}_R$, there exists a nonnegative local solution of (1) on an interval $[0, T_R]$, where $T_R > 0$ is dependent on R but uniform for $U_0 \in \mathcal{K}_R$. Then, we can indeed show that

$$(6) \quad t^{\frac{1+\varepsilon}{2}} \|U(t) - \widetilde{U}(t)\}\|_{H^{1+\varepsilon}} + \|U(t) - \widetilde{U}(t)\|_{L_2} \\ \leq L_R \|U_0 - \widetilde{U}_0\|_{L_2}, \qquad 0 \leq t \leq T_R;\ U_0, \widetilde{U}_0 \in \mathcal{K}_R,$$

where $U(t) = {}^t(u(t), v(t))$ and $\widetilde{U}(t) = {}^t(\widetilde{u}(t), \widetilde{v}(t))$ are local solutions of (1) for initial values U_0 and $\widetilde{U}_0$, respectively.

3. A priori estimates

We shall establish a priori estimates for local solutions with initial values from $\mathcal{K}$. Let $U_0 \in \mathcal{K}$ and let U denote any nonnegative local solution to Problem (1) in the function space

$$(7) \quad 0 \leq u,\ v \in \mathcal{C}((0, T_U]; H^2_N(\Omega)) \cap \mathcal{C}([0, T_U]; H^{1+\varepsilon}(\Omega)) \\ \cap\, \mathcal{C}^1((0, T_U]; H^1(\Omega)) \cap \mathcal{C}^2((0, T_U]; H^1(\Omega)^*),$$

where $[0, T_U]$ denotes the interval on which U is defined. As shown in the preceding sections, such a local solution exists at least on some interval $[0, T_{U_0}]$.

In this section the assumption (2) will essentially be used. But the techniques of proof are quite different by the cases when $\alpha_{12}\alpha_{21} > 0$ and when $\alpha_{12}\alpha_{21} = 0$.

3.1. **Case when $\boldsymbol{\alpha_{12}\alpha_{21} > 0}$.** Let us begin with noticing some scalling property. Let $\lambda > 0$ and $\mu > 0$ be two parameters and multiply the equations on u and v by λ and μ, respectively. Then we obtain an equivalent problem to (1):

$$\begin{cases} \dfrac{\partial u_\lambda}{\partial t} = \Delta(au_\lambda + \alpha_{11}\lambda^{-1}u_\lambda^2 + \alpha_{12}\mu^{-1}u_\lambda v_\mu) & \\ \qquad\qquad + cu_\lambda - \gamma_{11}\lambda^{-1}u_\lambda^2 - \gamma_{12}\mu^{-1}u_\lambda v_\mu & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial v_\mu}{\partial t} = \Delta(bv_\mu + \alpha_{21}\lambda^{-1}u_\lambda v_\mu + \alpha_{22}\mu^{-1}v_\mu^2) & \\ \qquad\qquad + dv_\mu - \gamma_{21}\lambda^{-1}u_\lambda v_\mu - \gamma_{22}\mu^{-1}v_\mu^2 & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial u_\lambda}{\partial n} = \dfrac{\partial v_\mu}{\partial n} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u_\lambda(x,0) = \lambda u_0(x), \quad v_\mu(x,0) = \mu v_0(x) & \text{in } \Omega, \end{cases}$$

where $u_\lambda(x,t) = \lambda u(x,t)$ and $v_\mu(x,t) = \mu v(x,t)$. It is clear that the condition (2) is invariant under such scaling.

If we choose λ and μ so that the relation $\alpha_{12}\lambda = \sqrt{8\alpha_{11}\alpha_{21}}\mu$ is valid, then it is observed that

$$(\alpha_{12}\mu^{-1})^2 = 8(\alpha_{11}\lambda^{-1})(\alpha_{21}\lambda^{-1}) \quad \text{and} \quad (\alpha_{21}\lambda^{-1})^2 \le 8(\alpha_{22}\mu^{-1})(\alpha_{12}\mu^{-1}).$$

So we verify that (2) is essentially equivalent to the stronger conditions

$$0 < \alpha_{12}^2 \le 8\alpha_{11}\alpha_{21} \quad \text{and} \quad 0 < \alpha_{21}^2 \le 8\alpha_{22}\alpha_{12}. \tag{8}$$

As a matter of fact, in this subsection, we will assume these relations.

Proposition 1. *There exists a continuous increasing function $p(\cdot)$ such that, for any local solution U in the space (7) with initial value $U_0 \in \mathcal{K} \cap H^2(\Omega)$, it holds that*

$$\|U(t)\|_{H^2} \le p(\|U_0\|_{H^2}), \qquad 0 \le t \le T_U.$$

We can carry out the proof of this proposition without any change as that of Proposition 4.2 in [3]. So, we will omit the proof.

3.2. **Case when $\boldsymbol{\alpha_{12}\alpha_{21} = 0}$.** Let us assume that one of α_{12} and α_{21} vanishes, say, $\alpha_{21} = 0$ and so $Q(u,v) = Q(v) = bv + \alpha_{22}v^2$.

Proposition 2. *Under the same situation as in Proposition 1, for any local solution U belonging to the space (7), it holds that*

$$\|U(t)\|_{H^2} \le p(\|U_0\|_{H^2}), \qquad 0 \le t \le T_U \tag{9}$$

with some suitable continuous increasing function $p(\cdot)$.

Proof. Step 1. Integrate the first equation of (1) in Ω. Then,

$$\frac{d}{dt}\int_\Omega u\,dx = \int_\Omega f(u,v)dx \le \int_\Omega (cu - \gamma_{11}u^2)dx.$$

Since

$$cu - \gamma_{11}u^2 \le -u + \tfrac{(c+1)^2}{4\gamma_{11}} \qquad \text{for all } u \ge 0,$$

we conclude that

(10) $$\|u(t)\|_{L_1} \le e^{-t}\|u_0\|_{L_1} + \tfrac{(c+1)^2}{4\gamma_{11}}|\Omega|, \qquad 0 \le t \le T_U.$$

In addition, from

$$\|u(t)\|_{L_1} - \|u_0\|_{L_1} = \int_0^t \int_\Omega f(u,v)dxds,$$

it follows that

(11) $$\left|\int_0^t \int_\Omega f(u,v)dxds\right| \le \|u_0\|_{L_1} + \tfrac{(c+1)^2}{4\gamma_{11}}|\Omega|, \qquad 0 \le t \le T_U.$$

Step 2. Let $2 < q < \infty$. Multiply the second equation of (1) by qv^{q-1} and integrate the product in Ω. Then,

$$\frac{d}{dt}\int_\Omega v^q dx = -q(q-1)\int_\Omega Q_v v^{q-2}|\nabla v|^2 dx + q\int_\Omega g v^{q-1}dx \le q\int_\Omega (d-\gamma_{22}v)v^q dx.$$

Noting that, for example,

$$(d-\gamma_{22}v)v^q \le -dv^q + \gamma_{22}(2d/\gamma_{22})^{q+1} \qquad \text{for all } 0 \le v < \infty$$

(divide the proof into two cases $0 \le v \le 2d/\gamma_{22}$ and $2d/\gamma_{22} < v < \infty$), we have

$$\frac{d}{dt}\int_\Omega v^q dx \le -dq\int_\Omega v^q dx + \gamma_{22}q(2d/\gamma_{22})^{q+1}|\Omega|.$$

Therefore,

$$\|v(t)\|_{L_q}^q \le e^{-dqt}\|v_0\|_{L_q}^q + 2(2d/\gamma_{22})^q|\Omega|$$

and

$$\|v(t)\|_{L_q} \le 2^{\frac{1}{q}}\{e^{-dt}\|v_0\|_{L_q} + (2d/\gamma_{22})(2|\Omega|)^{\frac{1}{q}}\}.$$

Since $\lim_{q\to\infty}\|v\|_{L_q} = \|v\|_{L_\infty}$, we obtain that

(12) $$\|v(t)\|_{L_\infty} \le e^{-dt}\|v_0\|_{L_\infty} + \tfrac{2d}{\gamma_{22}}, \qquad 0 \le t \le T_U.$$

As well, since

(13) $$u^2 = \gamma_{11}^{-1}\{cu - \gamma_{12}uv - f(u,v)\},$$

(12) jointed with (10) and (11) yields that

(14) $$\int_0^t \|u(s)\|_{L_2}^2 ds \le p(\|u_0\|_{L_1} + \|v_0\|_{L_\infty})(t+1), \qquad 0 \le t \le T_U.$$

Step 3. Next, multply the second equation of (1) by $\frac{dQ}{dt}$ and integrate the product in Ω. Then, in view of (12),

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla Q|^2 dx + \int_\Omega Q_v v_t^2 dx = \int_\Omega gQ_v v_t dx \le p(\|v_0\|_{L_\infty})\int_\Omega (u+v+1)|v_t|dx$$
$$\le \frac{b}{2}\int_\Omega v_t^2 dx + p(\|v_0\|_{L_\infty})\int_\Omega (u+v+1)^2 dx.$$

Therefore,

(15) $$\frac{d}{dt}\int_\Omega |\nabla Q|^2 dx + \int_\Omega bv_t^2 dx \le p(\|v_0\|_{L_\infty})\int_\Omega (u^2+1)dx.$$

In the meantime, multply the second equation by Q and integrate the product in Ω. Then, in view of (10),

$$\frac{d}{dt}\int_\Omega \Xi(v(t))dx + \int_\Omega |\nabla Q|^2 dx = \int_\Omega gQdx \le p(\|v_0\|_{L_\infty})\int_\Omega (u+1)dx$$
$$\le p(\|u_0\|_{L_1} + \|v_0\|_{L_\infty}),$$

where $\Xi(v) = \int_0^v Q(v)dv = \frac{b}{2}v^2 + \frac{\alpha_{22}}{3}v^3$. This differential inequality together with (15) then yields that

$$(16)\quad \frac{d}{dt}\int_\Omega (\Xi + |\nabla Q|^2)dx + \int_\Omega (\Xi + |\nabla Q|^2)dx + \int_\Omega bv_t^2 dx$$
$$\le p(\|u_0\|_{L_1} + \|v_0\|_{L_\infty})\int_\Omega (u^2+1)dx.$$

Therefore,

$$\|\Xi(v(t))\|_{L_1} + \|\nabla Q(v(t))\|_{L_2}^2 \le e^{-t}(\|\Xi(v_0)\|_{L_1} + \|\nabla Q(v_0)\|_{H^1}^2)$$
$$+ p(\|u_0\|_{L_1} + \|v_0\|_{L_\infty})\int_0^t e^{-(t-s)}\{\|u(s)\|_{L_2}^2 + 1\}ds.$$

In view of (11), we can here repeat the same argument as for Lemma ref9:L1 to conclude that

$$(17)\qquad \int_0^t e^{-(t-s)}\|u(s)\|_{L_2}^2 ds \le p(\|u_0\|_{L_1} + \|v_0\|_{L_\infty}), \qquad 0 \le t \le T_U.$$

Indeed, we verify from (13) that

$$\|u(t)\|_{L_2}^2 \le -\tfrac{1}{\gamma_{11}}\int_\Omega f(u(t), v(t))dx + p(\|u_0\|_{L_1} + \|v_0\|_{L_\infty}), \qquad 0 \le t \le T_U.$$

So, the desired estimate (17) is obtained.

In this way, we have concluded that

$$(18)\qquad \|Q(v(t))\|_{H^1} \le Ce^{-t}\|Q(v_0)\|_{H^1} + p(\|u_0\|_{L_1} + \|v_0\|_{L_\infty}), \qquad 0 \le t \le T_U,$$

as well as

$$(19)\qquad \int_0^t \|v_t\|_{L_2}^2 ds \le p(\|u_0\|_{L_1} + \|v_0\|_{L_\infty})(t+1), \qquad 0 \le t \le T_U.$$

Since $\nabla v = Q_v^{-1}\nabla Q$, it follows from (18) that

$$(20)\qquad \|v(t)\|_{H^1} \le Ce^{-t}\|Q(v_0)\|_{H^1} + p(\|u_0\|_{L_1} + \|v_0\|_{L_\infty}), \qquad 0 \le t \le T_U.$$

Step 4. In this Step, we shall use an abbreviated notaion

$$p_1(U_0) = p\big(\|u_0\|_{L_1} + \|v_0\|_{L_\infty} + \|Q(v_0)\|_{H^1}\big),$$

where $p(\cdot)$ is a continuous increasing functions which varies in each occurrence. Introducing a quantity

$$N_{1,\log}(u) = \int_\Omega u\log(u+1)dx, \qquad 0 \le u \in L_2(\Omega),$$

we intend to estimate $N_{1,\log}(u(t))$ for the local solution.

Multiply the first equation of (1) by $\log(u+1)$ and integrate the product in Ω. Then, we observe that

$$\frac{d}{dt}\int_\Omega\{(u+1)\log(u+1)-u\}dx+4\int_\Omega P_u|\nabla\sqrt{u+1}|^2dx$$
$$=-\alpha_{12}\int_\Omega\{\log(u+1)-u\}\Delta v\,dx+\int_\Omega f\log(u+1)dx.$$

Furthermore,

$$-\alpha_{12}\int_\Omega\{\log(u+1)-u\}\Delta v\,dx\le\zeta\|\Delta v\|_{L_2}^2+C_\zeta\|u\|_{L_2}^2$$

with any $\zeta>0$. Since

$$\Delta v=\frac{\Delta Q-2\alpha_{22}|\nabla v|^2}{b+2\alpha_{22}v}=\frac{\Delta Q-2\alpha_{22}Q_v^{-2}|\nabla Q|^2}{b+2\alpha_{22}v},\qquad 0\le v\in H^2(\Omega),$$

it follows by $\|u\|_{H^2}\le C(\|\Delta u\|_{L_2}+\|u\|_{L_2}$, $u\in H^2_N(\Omega)$, that

$$(21)\quad \|\Delta v\|_{L_2}^2\le C(\|\Delta Q\|_{L_2}^2+\|Q\|_{H_4^1}^4)$$
$$\le C\{\|\Delta Q\|_{L_2}^2(1+\|Q\|_{H^1}^2)+\|Q\|_{H^1}^4\},\qquad 0\le v\in H^2(\Omega).$$

Meanwhile, we have

$$C_\zeta\|u\|_{L_2}^2\le-\frac{1}{2}\int_\Omega(cu-\gamma_{11}u^2)\log(u+1)dx+\widetilde{C}_\zeta\|u\|_{L_1},\qquad 0\le u\in L_2(\Omega),$$

whatever the parameter ζ is, but $\widetilde{C}_\zeta$ denotes some constant determined from C_ζ (and hence from ζ). Hence, we obtain that

$$\frac{d}{dt}\int_\Omega\{(u+1)\log(u+1)-u\}dx+\int_\Omega\{(u+1)\log(u+1)-u\}dx$$
$$\le\zeta C\{\|\Delta Q\|_{L_2}^2(1+\|Q\|_{H^1}^2)+\|Q\|_{H^1}^4\}+\widetilde{C}_\zeta\|u\|_{L_1}.$$

Moreover, in view of (10) and (18),

$$\frac{d}{dt}\int_\Omega\{(u+1)\log(u+1)-u\}dx+\int_\Omega\{(u+1)\log(u+1)-u\}dx$$
$$\le p_1(U_0)(\zeta\|v_t-g\|_{L_2}^2+\widetilde{C}_\zeta).$$

We then add this to the differential inequality (16) and take ζ sufficiently small. Then, it follows that

$$\frac{d\psi_1}{dt}+\psi_1(t)+\frac{b}{2}\int_\Omega v_t^2dx\le p_1(U_0)(\|u\|_{L_2}^2+1),$$

where $\psi_1(t)=\int_\Omega\{(u+1)\log(u+1)-u+\Xi+|\nabla Q|^2\}dx$. Noting (17), we conclude that

$$\psi_1(t)\le e^{-t}\psi_1(0)+p_1(U_0),\qquad 0\le t\le T_U.$$

In particular,

$$(22)\qquad N_{1,\log}(u(t))\le e^{-t}N_{1,\log}(u_0)+p_1(U_0),\qquad 0\le t\le T_U.$$

Step 5. In this Step again, we shall use an abbreviated notaion

$$p_2(U_0) = p\big(N_{1,\log}(u_0) + \|v_0\|_{L_\infty} + \|Q(v_0)\|_{H^1}\big),$$

where $p(\cdot)$ is a continuous increasing function which varies in each occurrence.

Multiply the first equation of (1) by u and integrate the product in Ω. Then,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \int_\Omega P_u|\nabla u|^2 dx = -\int_\Omega P_v \nabla u\cdot\nabla v\,dx + \int_\Omega fu\,dx$$

$$= -\alpha_{12}\int_\Omega u\nabla u\cdot\nabla v\,dx + \int_\Omega fu\,dx.$$

Here,

$$-\int_\Omega u\nabla u\cdot\nabla v\,dx = -\frac{1}{2}\int_\Omega \nabla u^2\cdot\nabla v\,dx = \frac{1}{2}\int_\Omega u^2\Delta v\,dx$$

$$\le C\|u\|_{L_3}^2\|\Delta v\|_{L_3} \le \zeta_1\|\Delta v\|_{L_3}^3 + C_{\zeta_1}\|u\|_{L_3}^3$$

with any $\zeta_1 > 0$. Furthermore, by (20),

$$\|\Delta v\|_{L_3}^3 \le C\|\Delta v\|_{H^1}\|\Delta v\|_{L_2}^2 \le C\|v\|_{H^3}^2\|v\|_{H^1} \le p_2(U_0)\|v\|_{H^3}^2.$$

In addition, by eqref0:, (10) and (22),

$$\|u\|_{L_3}^3 \le \zeta_2\|u\|_{H^1}^2 N_{1,\log}(u) + C_{\zeta_2}\|u\|_{L_1} \le p_2(U_0)\{\zeta_2\|u\|_{H^1}^2 + C_{\zeta_2}\}$$

with any $\zeta_2 > 0$. Therefore, it follows that

$$\zeta_1\|\Delta v\|_{L_3}^3 + C_{\zeta_1}\|u\|_{L_3}^3 \le p_2(U_0)\{\zeta_1\|v\|_{H^3}^2 + C_{\zeta_1}\zeta_2\|u\|_{H^1}^2 + C_{\zeta_1}C_{\zeta_2}\}.$$

We hence obtain that

$$\text{(23)}\quad \frac{d}{dt}\int_\Omega u^2 dx + \gamma_{22}\int_\Omega u^3 dx + 2\int_\Omega P_u|\nabla u|^2 dx$$

$$\le p_2(U_0)\{\zeta_1\|v\|_{H^3}^2 + C_{\zeta_1}\zeta_2\|u\|_{H^1}^2 + C_{\zeta_1}C_{\zeta_2}\}.$$

In the meantime, we prepare another differential inequality

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\Delta v|^2 dx + \int_\Omega Q_v|\nabla\Delta v|^2 dx$$

$$= -\alpha_{22}\int_\Omega (\Delta v\nabla v + \nabla|\nabla v|^2)\cdot\nabla\Delta v\,dx - \int_\Omega \nabla g\cdot\nabla\Delta v\,dx$$

$$\le p_2(U_0)\|v\|_{H^3}\{\|v\|_{H^2_3}\|v\|_{H^1_6} + \|\nabla u\|_{L_2} + 1\}.$$

This equation is indeed obtained by considering the scaler product of the second equation of (1) with $\Delta^2 v$ in $H^1(\Omega)\times H^1(\Omega)^*$. Note that, since Ω is of class $\mathcal{C}^3$, $\Delta Q \in H^1(\Omega)$ with $\frac{\partial Q}{\partial n} = 0$ on $\partial\Omega$ implies $Q \in H^3(\Omega)$ with the estimate $\|Q\|_{H^3} \le C\{\|\nabla\Delta Q\|_{L_2} + \|Q\|_{H^1}\}$. Therefore, we have $v \in \mathcal{C}^1((0,\infty); H^3(\Omega))$. Note also that Δ^2 is a bounded operator from $H^3(\Omega)$ into $H^1(\Omega)^*$.

Here,

$$\|v\|_{H^3}\|v\|_{H^2_3}\|v\|_{H^1_6} \le C\|v\|_{H^3}^{\frac{4}{3}}\|v\|_{H^2}^{\frac{4}{3}}\|v\|_{H^1}^{\frac{1}{3}} \le \zeta_3\|v\|_{H^3}^2 + C_{\zeta_3}p_2(U_0)\|v\|_{H^2}^4$$

with any $\zeta_3 > 0$. And,

$$\|v\|_{H^3}\|\nabla u\|_{L_2} \le \zeta_4\|v\|_{H^3}^2 + C_{\zeta_4}\|\nabla u\|_{L_2}^2$$

with any $\zeta_4 > 0$. So, taking ζ_3 and ζ_4 sufficiently small, we obtain that

$$\frac{d}{dt}\int_\Omega |\Delta v|^2 dx + 2\int_\Omega Q_v|\nabla\Delta v|^2 dx \le p_2(U_0)\{\|\nabla u\|_{L_2}^2 + \|v\|_{H^2}^4 + 1\}. \tag{24}$$

We multiply a parameter $\eta > 0$ to the inequality (23) and add the multiplied inequality to (24). Then,

$$\frac{d}{dt}\int_\Omega (\eta u^2 + |\Delta v|^2)dx + \gamma_{22}\eta\int_\Omega u^3 dx + \int_\Omega [\{2a\eta - p_2(U_0)\}|\nabla u|^2 \\ + 2b|\nabla\Delta v|^2]dx \le \eta p_2(U_0)\{\zeta_1\|v\|_{H^3}^2 + C_{\zeta_1}\zeta_2\|u\|_{H^1}^2 + C_{\zeta_1}C_{\zeta_2}\} + p_2(U_0)\|v\|_{H^2}^4.$$

If η is fiexed in such a way that $a\eta \ge p_2(U_0)$ and if ζ_1 and then ζ_2 are taken sufficiently small, then it is verified that

$$\frac{d}{dt}\int_\Omega (\eta u^2 + |\Delta v|^2)dx + \gamma_{22}\eta\int_\Omega u^3 dx + \int_\Omega (a\eta|\nabla u|^2 + b|\nabla\Delta v|^2)dx \\ \le p_2(U_0)(\|v\|_{H^2}^4 + 1).$$

Moreover, in view of (21),

$$\|v\|_{H^2}^4 \le C(\|\Delta v\|_{L_2}^4 + \|v\|_{L_2}^4) \le p_2(U_0)(\|\Delta v\|_{L_2}^2\|\Delta Q\|_{L_2}^2 + 1) \\ \le p_2(U_0)\{\|\Delta v\|_{L_2}^2(\|v_t\|_{L_2}^2 + \|u\|_{L_2}^2 + 1) + 1\}.$$

Hence,

$$\frac{d}{dt}\int_\Omega (\eta u^2 + |\Delta v|^2)dx + \gamma_{22}\eta\int_\Omega u^3 dx + \int_\Omega (a\eta|\nabla u|^2 + b|\nabla\Delta v|^2)dx \\ \le p_2(U_0)\{\|\Delta v\|_{L_2}^2(\|u\|_{L_2}^2 + \|v_t\|_{L_2}^2 + 1) + 1\}.$$

We add this inequality to the following one

$$\frac{d}{dt}\int_\Omega \xi|\nabla Q|^2 dx + \int_\Omega \xi p(U_0)^{-1}|\Delta v|^2 dx \le p_2(U_0)\xi(\|u\|_{L_2}^2 + 1)$$

which is obtained from (15) (due to (21)) after some calculations and multiplication of a parameter $\xi > 0$. Then,

$$\frac{d}{dt}\psi_2(t) + \xi p_2(U_0)^{-1}\psi_2(t) + \int_\Omega (a\eta|\nabla u|^2 + b|\nabla\Delta v|^2)dx \\ \le p_2(U_0)[\{\|u(t)\|_{L_2}^2 + \|v_t(t)\|_{L_2}^2 + 1\}\psi_2(t) + C_\xi], \tag{25}$$

where $\psi_2(t) = \eta\|u(t)\|_{L_2}^2 + \xi\|\nabla Q(t)\|_{L_2}^2 + \|\Delta v(t)\|_{L_2}^2$. Here, we used a fact that the estimate

$$\{p_2(U_0) + p_2(U_0)^{-1}\}\xi u^2 \le \gamma_{22}\eta u^3 + C_\xi p_2(U_0) \qquad \text{for all } u \ge 0$$

holds, whatever the parameter $\xi > 0$ is, with some constant $C_\xi > 0$.

Solving (25), we conclude that

$$\psi_2(t) \le e^{\int_0^t [p_2(U_0)\{\|u(s)\|_{L_2}^2 + \|v_s(s)\|_{L_2}^2 + 1\} - p_2(U_0)^{-1}\xi]ds}\psi_2(0) \\ + C_\xi p_2(U_0)\int_0^t e^{\int_s^t [p_2(U_0)\{\|u(\tau)\|_{L_2}^2 + \|v_\tau(\tau)\|_{L_2}^2 + 1 - p_2(U_0)^{-1}\xi]d\tau}ds.$$

Moreover, by (14) and (19) (especially applying these estimates by substituting s for 0 and substituting $u(s)$ and $v(s)$ for u_0 and v_0, respectively),

$$\psi_2(t) \le p_2(U_0)e^{\{p_2(U_0)-p_2(U_0)^{-1}\xi\}t}\psi(0) + C_\xi p_2(U_0)\int_0^t e^{\{p_2(U_0)-p_2(U_0)^{-1}\xi\}(t-s)}ds.$$

It is here possible to fix the parameter ξ in such a way that $p_2(U_0) - p_2(U_0)^{-1}\xi \le -1$. It then follows that $\psi_2(t) \le p_2(U_0)\{e^{-t}\psi_2(0)+1\}$. In particular,

$$\|u(t)\|_{L_2}^2 + \|\Delta v(t)\|_{L_2}^2 \le p_2(U_0)\{e^{-t}(\|u_0\|_{L_2}^2 + \|v_0\|_{H^2}^2)+1\}, \qquad 0 \le t \le T_U.$$

In view of (20),

$$\|u(t)\|_{L_2}^2 + \|v(t)\|_{H^2}^2 \le p_2(U_0)\{e^{-t}(\|u_0\|_{L_2}^2 + \|v_0\|_{H^2}^2)+1\}, \qquad 0 \le t \le T_U. \tag{26}$$

It is as well obtained from (25) that

$$\int_0^t \{\|u(s)\|_{H^1}^2 + \|v(s)\|_{H^3}^2\}ds \le p(\|u_0\|_{L_2} + \|v_0\|_{H^2})(t+1), \qquad 0 \le t \le T_U. \tag{27}$$

Step 6. In this Step, we shall use a notation

$$p_3(U_0) = p(\|u_0\|_{L_2} + \|v_0\|_{H^2}),$$

where $p(\cdot)$ is a similar continuous increasing function as before.

Multiply the first equation of (1) by u^3 and integrate the product in Ω. Then,

$$\frac{1}{4}\frac{d}{dt}\int_\Omega u^4dx + \int_\Omega 3P_u u^2|\nabla u|^2dx = -\int_\Omega 3u^2P_v\nabla u\cdot\nabla v dx + \int_\Omega fu^3dx.$$

Here,

$$\int_\Omega u^3|\nabla u||\nabla v|dx \le \int_\Omega u^2|u\nabla u|\|\nabla v\|_{L_\infty}dx$$
$$\le \int_\Omega (\zeta u^2|\nabla u|^2 + C_\zeta\zeta'\|\nabla v\|_{L_\infty}^{12} + C_\zeta C_{\zeta'}u^{\frac{24}{5}})dx$$

with any $\zeta > 0$ and $\zeta' > 0$. In addition, $\|\nabla v\|_{L_\infty} \le \|\nabla v\|_{H^{13/12}} \le C\|v\|_{H^3}^{1/12}\|v\|_{H^2}^{11/12}$. Therefore, if the parameter ζ is fixed sufficiently small, then

$$\frac{d}{dt}\int_\Omega u^4dx + \int_\Omega u^4dx + \int_\Omega au^2|\nabla u|^2dx \le p_3(U_0)\{\zeta'\|v\|_{H^3}^2 + \widetilde{C}_{\zeta'}\}.$$

We here used a fact that

$$C_{\zeta'}u^{\frac{24}{5}} + u^4 + cu \le \gamma_{11}u^5 + \widetilde{C}_{\zeta'}, \qquad u \ge 0,$$

whatever the parameter $\zeta' > 0$ is, where $\widetilde{C}_{\zeta'} > 0$ is another constant depending on ζ'.

We add this differential inequality to (25). Taking ζ' sufficiently small, we obtain that

$$\frac{d}{dt}\psi_3(t) + \delta_1\psi_3(t) + \int_\Omega au^2|\nabla u|^2dx \le p_3(U_0),$$

where $\psi_3(t) = \psi_2(t) + \|u(t)\|_{L_4}^4$, with some positive exponent $\delta_1 > 0$. Hence,

$$\psi_3(t) \le p_3(U_0)\{e^{-\delta_1 t}\psi_3(0)+1\}, \qquad 0 \le t \le T_U.$$

In particular,

$$\|u(t)\|_{L_4}^4 \le p_3(U_0)\{e^{-\delta_1 t}\|u_0\|_{L_4}^4 + 1\}, \qquad 0 \le t \le T_U. \tag{28}$$

Step 7. We shall use the following notation

$$p_4(U_0) = p(\|u_0\|_{L_4} + \|v_0\|_{H^2}).$$

Our goal of the present step is to estimate the norm $\|u(t)\|_{H^1}$ for the local solution.

Multiply the first equation of (1) by $\frac{d}{dt}P(u,v) = P_u u_t + P_v v_t$ and integrate the product in Ω. Then,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla P|^2 dx + \int_\Omega P_u u_t^2 dx = -\int_\Omega P_v u_t v_t dx$$
$$+\int_\Omega f\{P_u u_t + P_v v_t\}dx \le C\int_\Omega \{u|u_t||v_t| + (u^3+v^3)(|u_t|+|v_t|)\}dx.$$

Here, in view of (28),

$$\int_\Omega u|u_t||v_t|dx \le \|u\|_{L_4}\|u_t\|_{L_2}\|v_t\|_{L_4} \le C\|u\|_{L_4}\|u_t\|_{L_2}\|v_t\|_{H^1}^{\frac{1}{2}}\|v_t\|_{L_2}^{\frac{1}{2}}$$
$$\le p_4(U_0)\|u_t\|_{L_2}\|\Delta Q + g\|_{H^1}^{\frac{1}{2}} \le p_4(U_0)\|u_t\|_{L_2}(\|v\|_{H^3} + \|u\|_{H^1}+1)^{\frac{1}{2}}$$
$$\le \tfrac{a}{8}\|u_t\|_{L_2}^2 + p_4(U_0)(\|u\|_{H^1} + \|v\|_{H^3}+1).$$

Similarly,

$$\int_\Omega u^3|u_t|dx \le \|u\|_{L_6}^3\|u_t\|_{L_2} \le \|u\|_{H^1}\|u\|_{L_4}^2\|u_t\|_{L_2}$$
$$\le p_4(U_0)\|u\|_{H^1}\|u_t\|_{L_2} \le \tfrac{a}{8}\|u_t\|_{L_2}^2 + p_4(U_0)\|u\|_{H^1}^2$$

and

$$\int_\Omega u^3|v_t|dx \le \|u\|_{L_6}^3\|v_t\|_{L_2} \le p_4(U_0)\|u\|_{H^1}.$$

Furthermore, it is immediate from $\nabla P(u,v) = P_u\nabla u + P_v\nabla v$ to see that

$$\|\nabla u\|_{L_2} \le C\{\|\nabla P\|_{L_2} + \|u\|_{L_4}\|v\|_{H^2}\}, \qquad 0\le u,\ v\in H^2(\Omega). \tag{29}$$

In addition,

$$\|\nabla P\|_{L_2}^2 \le \zeta\|\Delta P\|_{L_2}^2 + C_\zeta\|P\|_{L_2}^2, \qquad P\in H_N^2(\Omega).$$

with any $\zeta > 0$. Hence, it is obtained that

$$\frac{d}{dt}\int_\Omega |\nabla P|^2dx + \int_\Omega |\nabla P|^2 dx + \int_\Omega a|\Delta P|^2 dx \le p_4(U_0)(\|v\|_{H^3}+1)$$
$$\le \zeta'\|v\|_{H^3}^2 + C_{\zeta'}p_4(U_0)$$

with any $\zeta' > 0$.

We add this differential inequality to (25). If the parameter ζ' is fixed sufficiently small, then

$$\frac{d}{dt}\psi_4(t) + \delta_2\psi_4(t) + \int_\Omega a|\Delta P|^2dx \le p_4(U_0), \tag{30}$$

where $\psi_4(t) = \psi_2(t) + \|\nabla P(t)\|_{L_2}^2$, with some positive exponent $\delta_2 > 0$. Hence,

$$\|P(u(t),v(t))\|_{H^1}^2 \le p_4(U_0)\{e^{-\delta_2 t}\|P(u_0,v_0)\|_{H^1}^2 + 1\}, \qquad 0\le t\le T_U. \tag{31}$$

By (29),

$$\|u(t)\|_{H^1}^2 \le p_4(U_0)\{e^{-\delta_2 t}\|P(u_0,v_0)\|_{H^1}^2+1\}, \qquad 0\le t\le T_U. \tag{32}$$

It is as well obtained that

$$\int_0^t \|P\|_{H^2}^2 ds \le p\big(\|P(u_0,v_0)\|_{H^1}+\|v_0\|_{H^2}\big)(t+1), \qquad 0\le t\le T_U. \tag{33}$$

Step 8. We shall use the following notation

$$p_5(U_0)=p\big(\|P(u_0,v_0)\|_{H^1}+\|v_0\|_{H^2}\big).$$

We will use (5). Namely, differentiate the first equation of (1) in t and consider the scaler product with u_t in $H^1(\Omega)^*\times H^1(\Omega)$. From (5) it follows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega u_t^2dx+\int_\Omega P_u|\nabla u_t|^2 \le \alpha_{12}\int_\Omega u|\nabla u_t||\nabla v_t|dx + C\int_\Omega(|\nabla u|+|\nabla v|)\\ \times(|u_t|+|v_t|)|\nabla u_t|dx + C\int_\Omega (u+v+1)(u_t^2+v_t^2)dx.\end{aligned}$$

Here,

$$\int_\Omega u|\nabla u_t||\nabla v_t|dx \le \|u\|_{L_\infty}\|\nabla u_t\|_{L_2}\|\nabla v_t\|_{L_2} \le \zeta\|\nabla u_t\|_{L_2}^2 + C_\zeta\|u\|_{H^2}^2\|\nabla v_t\|_{L_2}^2$$

with any $\zeta>0$. In addition, by Lemma 1 which is presented below,

$$\|u\|_{H^2}\le p_5(U_0)(\|\Delta P\|_{L_2}+1)\le p_5(U_0)(\|u_t\|_{L_2}+1)$$

and, by a direct calculation,

$$\|\nabla v_t\|_{L_2}\le \|\nabla(\Delta Q+g)\|_{L_2}\le p_5(U_0)(\|v\|_{H^3}+1).$$

Similarly, by Lemma 1,

$$\begin{aligned}\int_\Omega(|\nabla u|+|\nabla v|)(|u_t|+|v_t|)|\nabla u_t|dx &\le (\|\nabla u\|_{L_4}+\|\nabla v\|_{L_4})(\|u_t\|_{L_4}+\|v_t\|_{L_4})\|\nabla u_t\|_{L_2}\\ &\le (\|u\|_{H^2}^{\frac12}\|u\|_{H^1}^{\frac12}+\|v\|_{H^2}^{\frac12}\|v\|_{H^1}^{\frac12})(\|u_t\|_{H^1}^{\frac12}\|u_t\|_{L_2}^{\frac12}+\|v_t\|_{H^1}^{\frac12}\|v_t\|_{L_2}^{\frac12})\|\nabla u_t\|_{L_2}\\ &\le p_5(U_0)(\|u_t\|_{L_2}^{\frac12}+1)(\|\nabla u_t\|_{L_2}^{\frac12}\|u_t\|_{L_2}^{\frac12}+\|u_t\|_{L_2}+\|v\|_{H^3}^{\frac12}+1)\|\nabla u_t\|_{L_2}\\ &\le \zeta\|\nabla u_t\|_{L_2}^2 + C_\zeta p_5(U_0)\|u_t\|_{L_2}^2(\|u_t\|_{L_2}^2+\|v\|_{H^3}^2+1)\end{aligned}$$

with any $\zeta>0$. Finally,

$$\int_\Omega(u+v+1)(u_t^2+v_t^2)dx \le \|u+v+1\|_{L_\infty}(\|u_t\|_{L_2}^2+\|v_t\|_{L_2}^2)\le p_5(U_0)(\|u_t\|_{L_2}^3+1).$$

Hence, it is verified that

$$\frac{d}{dt}\int_\Omega u_t^2dx+\int_\Omega a|\nabla u_t|^2dx \le p_5(U_0)(\|u_t\|_{L_2}^2+1)(\|P\|_{H^2}^2+\|v\|_{H^3}^2+1).$$

We add this differential inequality to (30) after multiplying both sides of(30) by a parameter $\xi > 0$. Then,

$$\frac{d}{dt}\psi_5(t) + a\xi\psi_5(t) + \int_\Omega a|\nabla u_t|^2 dx \\ \leq p_5(U_0)\{(\|P(u(t), v(t))\|_{H^2}^2 + \|v(t)\|_{H^3}^2 + 1)\psi_5(t) + C_\xi\},$$

where $\psi_5(t) = \|u_t(t)\|_{L_2}^2 + 1 + \xi\psi_4(t)$. Solving this differential inequality, we obtain that

$$\psi_5(t) \leq e^{\int_0^t [p_5(U_0)\{\|P(u(s),v(s))\|_{H^2}^2 + \|v(s)\|_{H^3}^2 + 1\} - a\xi]ds}\psi_5(0) \\ + C_\xi p_5(U_0)\int_0^t e^{\int_s^t [p_5(U_0)\{\|P(u(\tau),v(\tau))\|_{H^2}^2 + \|v(\tau)\|_{H^3}^2 + 1\} - a\xi]d\tau} ds.$$

Moreover, by (27) and (33) (which are used by substituting s for 0 and substituting $u(s)$ and $v(s)$ for u_0 and v_0, respectively),

$$\psi_5(t) \leq p_5(U_0)e^{\{p_5(U_0) - a\xi\}t}\psi_5(0) + C_\xi p_5(U_0)\int_0^t e^{\{p_5(U_0) - a\xi\}(t-\tau)} ds.$$

If the parameter ξ is fixed in such a way that $p_5(U_0) - a\xi \leq -1$, then

$$\psi_5(t) \leq p_5(U_0)\{e^{-t}\|u_0\|_{H^2}^2 + 1\}, \qquad 0 \leq t \leq T_U.$$

Hence, by Lemma 1 again, we conclude that

$$\|u(t)\|_{H^2}^2 \leq p_5(U_0)\{e^{-t}\|u_0\|_{H^2}^2 + 1\}, \qquad 0 \leq t \leq T_U. \tag{34}$$

It is as well obtained that

$$\int_0^t \|\nabla u_t\|_{L_2}^2 ds \leq p(\|u_0\|_{H^2} + \|v_0\|_{H^2})(t+1), \qquad 0 \leq t \leq T_U.$$

Thus, we have established the desired estimate (9). □

Lemma 1. *For $0 \leq u,\ v \in H^2(\Omega)$ with $\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0$ on $\partial\Omega$, it holds that*

$$\|u\|_{H^2} \leq C(\|P\|_{H^1}^2 + \|v\|_{H^2}^2 + 1)(\|\Delta P\|_{L_2} + \|P\|_{L_2}). \tag{35}$$

Proof of Lemma. We know that $\|u\|_{H^2} \leq C(\|\Delta u\|_{L_2} + \|u\|_{L_2})$. By a direct calculation, we see that

$$\Delta u = P_u^{-1}\big[\Delta P - P_{uu}P_u^{-2}|\nabla P|^2 + 2P_u^{-1}\{P_{uu}P_vP_u^{-1} - P_{uv}\}\nabla P\cdot\nabla v \\ + P_vP_u^{-1}\{2P_{uv} - P_{uu}P_vP_u^{-1}\}|\nabla v|^2\big].$$

Then the desired estimate (35) is verifed by the similar arguments as in the proof of Lemma 4.3 in [3]. □

Let us apply (10) and (12) in $[0, \frac{t}{7}]$, (18) in $[\frac{t}{7}, \frac{2t}{7}]$, (22) in $[\frac{2t}{7}, \frac{3t}{7}]$, (26) in $[\frac{3t}{7}, \frac{4t}{7}]$, (28) in $[\frac{4t}{7}, \frac{5t}{7}]$, (32) in $[\frac{5t}{7}, \frac{6t}{7}]$ and (34) in $[\frac{6t}{7}, t]$, respectively. Then we verify that

$$\|U(t)\|_{H^2} \leq p\big(e^{-\delta t}p(\|U_0\|_{H^2}) + 1\big), \qquad 0 \leq t \leq T_U \tag{36}$$

choosing some continuous increasing functions $p(\cdot)$'s and an exponent $\delta > 0$ suitably.

4. Global solutions

By Proposition 1 and Proposition 2, the global existence of solution is deduced. Indeed, we know that, for any initial value $U_0 \in \mathcal{K}$, there exists a nonnegative local solution at least on some interval $[0, T_U]$. Let $0 < t_1 < T_U$. Then, $U_1 \in [H^2_N(\Omega)]^2$. We next consider the problem (1) but with the initial value U_1 in the ball K_R of Z, where $R = p(\|U_1\|_{H^2})$. Proposition 1 when $\alpha_{12}\alpha_{21} > 0$ and Proposition 2 when $\alpha_{12}\alpha_{21} = 0$ ensure that any local solution starting from U_1 stays at any time in K_R. In addition, any local solution U on $[0, T_U]$ starting from U_1 can be extended over an interval $[0, T_U + \tau]$ as local solution, $\tau > 0$ being dependent only on $\sup_{0 \le t \le T_U} \|U(t)\|_{H^{1+\varepsilon}}$ and hence being independent of the extreme time T_U. This means that the problem (1) with the initial value U_1 possesses a unique global solution on the whole interval $[0, \infty)$.

We have thus deduced that, for any initial value $U_0 \in \mathcal{K}$, there exists a unique global solution to (1) in the function space

$$(37)\quad 0 \le u,\, v \in \mathcal{C}((0,\infty); H^2_N(\Omega)) \cap \mathcal{C}([0,\infty); H^{1+\varepsilon}(\Omega)) \\ \cap\, \mathcal{C}^1((0,\infty); H^{2\theta}(\Omega)) \cap \mathcal{C}^2((0,\infty); H^1(\Omega)^*), \qquad 0 \le \theta < 1.$$

It is as well obtained that, there exist continuous increasing functions $p(\cdot)$'s and a positive exponent $\delta > 0$ such that

$$(38)\qquad \|U(t)\|_{H^2} \le p\big(t^{\frac{\varepsilon-1}{2}} e^{-\delta t} p(\|U_0\|_{H^{1+\varepsilon}}) + 1\big), \qquad 0 < t < \infty;\ U_0 \in \mathcal{K}$$

are valid for all global solutions. As well,

$$(39)\qquad \|U(t)\|_{H^{1+\varepsilon}} \le p(\|U_0\|_{H^{1+\varepsilon}}), \qquad 0 < t < \infty;\ U_0 \in \mathcal{K}.$$

5. Dynamical system

5.1. **Construction of dynamical system.** Let us construct a dynamical system from the problem (1) on $\mathcal{K}$ in the universal space $L_2(\Omega)$. We already know that, for any $U_0 \in \mathcal{K}$, (1) possesses a unique global solution $U(t; U_0)$ in the function space (37). By $S(t)U_0 = U(t; U_0)$, we define a nonlinear semigroup $S(t)$ acting on $\mathcal{K}$.

As for the continuity of $S(t)$, we only know that $S(t)$ is continuous with respect to the L_2 topology in each bounded subset of $\mathcal{K}$, namely, by (6),

$$(40)\quad \|S(t)U_0 - S(t)V_0\|_{L_2} \le L^{n+1}_{p(\tilde{R})} \|U_0 - V_0\|_{L_2}, \\ t \in [n\tau_{p(\tilde{R})}, (n+1)\tau_{p(\tilde{R})});\ U_0,\, V_0 \in \mathcal{K}_{\tilde{R}};\ n = 0, 1, 2, \ldots.$$

Indeed, when $n = 0$, this is observed directly from (6). For general n, this is proved by induction on n in view of the fact (39).

We are then led to introduce a subset

$$\mathcal{X}_{\tilde{R}} = \bigcup_{U_0 \in \mathcal{K}_{\tilde{R}}} \{S(t)U_0;\ 0 \le t < \infty\}, \qquad 0 < \tilde{R} < \infty.$$

Obviously, $\mathcal{X}_{\tilde{R}}$ is an invariant set of $S(t)$ and $\mathcal{X}_{\tilde{R}} \subset \mathcal{K}_{p(\tilde{R})}$ due to (39). By the same reason as for (40), we have

$$\|S(t)U_0 - S(t)V_0\|_{L_2} \leq L_{p(\tilde{R})}^{n+1}\|U_0 - V_0\|_{L_2},$$
$$t \in [n\tau_{p(\tilde{R})}, (n+1)\tau_{p(\tilde{R})});\ U_0,\ V_0 \in \mathcal{X}_{\tilde{R}};\ n = 0, 1, 2, \ldots.$$

This together with the fact that $S(\cdot)U_0 \in \mathcal{C}([0,\infty); X)$ yields that $S(t)$ is a continuous semigroup on $\mathcal{X}_{\tilde{R}}$ in the topology of $L_2(\Omega)$.

Thus, Problem (1) determines a dynamical system $(S(t), \mathcal{X}_{\tilde{R}}, L_2(\Omega))$ for each $0 < \tilde{R} < \infty$.

5.2. **Compact absorbing set.** We shall show existence of a compact absrobing set for the semigroup $S(t)$.

We notice that (38) implies the following fact. For any $0 < r < \infty$, there exists a time $t_r > 0$ such that

$$S(t)\big(\mathcal{K} \cap \overline{B}^{H^{1+\varepsilon}}(0; r)\big) \subset \mathcal{K} \cap \overline{B}^{H^2}\big(0; p(1) + 1\big) \qquad \text{for all } t \geq t_r.$$

These in fact show that the set $\mathcal{B} = \mathcal{K} \cap \overline{B}^{H^2}(0; p(1) + 1)$ is an absorbing set of $S(t)$, namely, for any bounded subset B of $\mathcal{K}$, there exists a time t_B such that $S(t)B \subset B$ for all $t \geq t_B$. Obviously, $\mathcal{B}$ is a bounded subset of $H^2(\Omega)$ and is a compact subset of $L_2(\Omega)$.

5.3. **Exponential attractors.** We set

$$\mathcal{X} = \overline{\bigcup_{t_{\mathcal{B}} \leq t < \infty} S(t)\mathcal{B}} \subset \mathcal{B} \qquad \text{(closure in the space } L_2(\Omega)),$$

where $t_{\mathcal{B}} > 0$ is a time on which $\mathcal{B}$ is absorbed into itself. Then, $\mathcal{X}$ is seen to be an absorbing and invariant set of $S(t)$. Therefore, the asymptotic behavior of any dynamical system $(S(t), \mathcal{X}_{\tilde{\mathcal{R}}}, L_2(\Omega))$ is reduced to that of the smaller dynamical system $(S(t), \mathcal{X}, L_2(\Omega))$ in which $\mathcal{X}$ is a compact subset of $L_2(\Omega)$ and is a bounded subset of $H^2(\Omega)$.

In order to construct exponential attractors for $(S(t), \mathcal{X}, L_2(\Omega))$, it is now ready to follow the general procedure which has been described in Section 5 of [1]. Hence, we can construct a family of exponential attractors.

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Department of Applied Physics, Osaka University, Suita, Osaka 565-0871, Japan