Transversality Theorems and Knots in Three-Manifolds

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0. Introduction

In this paper we shall work in the smooth category for manifolds and mappings. Let M^n and V^{2n} be an n-manifold and a $2n$-manifold respectively and

$$f : M^n \longrightarrow V^{2n}$$

a smooth map. Suppose that $\{f^{-1}(q)\} = \{p_1, p_2\}$ consists of two different points for a point $q \in V^{2n}$ and that

$$T_{p_1} f(T_{p_1}(M^n)) \oplus T_{p_2} f(T_{p_2}(M^n)) = T_p(V^{2n}).$$

Then f is said to intersect transversely at q.

Now let $f : M^n \longrightarrow V^{2n}$ be an immersion as the notations above. Suppose that the following conditions are satisfied for any point $q \in f(M^n)$

(i) $f^{-1}(q)$ consists of only one point,

(ii) f intersects transversely at q.

If either (i) or(ii) is satisfied for f and any point $q \in f(M^n)$, then f is called a complete immersion.

Now let X, Y be metric spaces and $\delta : X \longrightarrow (0, \infty)$ a positive real-valued continuous function. For two continuous functions $f, g : X \longrightarrow Y$,

$$\rho(f(x), g(x)) < \delta(x)$$

is satisfied for any $x \in X$ where ρ is a metric of Y, then g is said a δ-approximation of f.

Now the following main theorem will be proved in the section 2 :

Theorem 2. 3. *Let M^3 be a connected 3-manifold with $\partial M^3 = S^2$ and*

$$f, g : I \longrightarrow M^3$$

proper embeddings such that f is homotopic to g rel. $f \mid [0, 1/3] = g \mid [0, 1/3]$. *And let*

$$F : I \times R \longrightarrow M^3 \times R$$

be a homotopy between f, g such that

$$F(s, t) = f(s) \quad \textit{for} \quad t \leqq 1/3$$
$$F(s, t) = g(s) \quad \textit{for} \quad 2/3 \leqq t$$

and

$$F(s, t) = f(s) = g(s) \quad \textit{for} \quad s \in [0, 1/3] \cup [2/3, 1].$$

Then there is a complete immersion

$$G : I \times R \longrightarrow M^3 \times R$$

such that the following conditions are satisfied

(1) *G is level-preserving*

(2) *G is a δ-approximation of F*

(3) *G is homotopic to F* rel. $I \times (R - \mathrm{Int}\, I) \cup N \times R$

where $N = [0, 1/3] \cup [2/3, 1] \subset I$

(4) $G : I \times R \longrightarrow M^3 \times R$ *is a regular homotopy between f and g.*

Now let N^3 be a simply-connected, closed 3-manifold and $f : S^1 \longrightarrow N^3$ be an embedding. Then $f(S^1) = K$ is said a knot in N^3. Suppose that there is a self-transverse immersion

$$f : D^2 \longrightarrow N^3$$

such that the following conditions are satisfied :

(1) $f \mid \partial D^2 : \partial D^2 \longrightarrow K$ is a diffeomorphism,

(2) $\{f^{-1}(q)\}$ consists of two points at most for any $q \in N^3$, and if $\{f^{-1}(q)\}$ consists of just two points, then q is called *a singular point*,

(3) Each component of all the singular points for f consists of an arc and there are two kinds of arcs, among these arcs, each of which is called a clasp arc or a ribbon arc. The inverse image of an arc by f consists of two arcs in D^2. The map

$$f : D^2 \longrightarrow N^3$$

has transverse intersections along these arcs. If one boundary point of each of the two arcs is contained into ∂D^2 and all other points of each arc are contained into $\mathrm{Int}D^2$, then the arc, which is the image of these two arcs in D^2 by the map f, is said a clasp arc. And If the two boundary points of an arc among the two arcs are contained into ∂D^2 and all other points of this arc and another arc are contained into $\mathrm{Int}D^2$, then the arc, which is the image of these two arcs in D^2 by the map f, is said a ribbon arc. Now for the transverse immersion $f : D^2 \longrightarrow N^3$ which satisfies the conditions above (1), (2), and (3), $f(D^2)$ is called *a singular 2-disk* of K.

Now at first referring to [4], for simplicity let N^3 be an orientable closed 3-manifold and suppose that N^3 has a triangulation. Let $f: S \longrightarrow N^3$ be a map such that S is a triangulated orientable surface of type (p, r), *i. e.*, an orientable compact surface of genus p with r boundary curves, and $f(\sigma)$ is a rectilinear 2-simplex in a 3-simplex of N^3, for any 2-simplex σ of S. Then f is called *singular locally-linear map*, and $f(S)$ is called a singular orientable surface of type (p, r). A singular orientable surface $f(S)$ of type (p, r) is called *normal*, if its singularities consist of *double curves* along which two sheets cross, *triple points* at which 3 sheets cut, and *branch points*. A point b is called a branch point if $f(S)$ cuts a small sphere with b as its center in a single non-simple curve. Let b be a branch point or a regular point. And let B^2 be a sufficiently small convex 2-sphere containing b in its interior, and let $K = f(S) \cap B^2$. This is a normal closed curve on B^2 which means that the closed curve has finite double points at most, the number of whose double points is called the multiplicity of b, and is denoted by $m(b)$. Clearly $m(b)$ does not depend on B^2. If $m(b) > 0$, then b is a branch point of $f(S)$, and if $m(b) = 0$, then b is a regular point of $f(S)$. By a Dehn surface of type (p, r) we mean a singular orientable surface of type (p, r) such that no point of the boundary of it is singular, *i. e.* if C is a component of its boundary, then there is a small tube T around C such that T intersects the surface in an annulus. A Dehn surface of type (p, r) is called *canonical* if it is normal and has no branch point. If $m(b) > 0$, by making a small cut of $f(S)$ starting at b and ending $b_1 \in B^2$ along a double curve, a new normal Dehn surface F of type (p, r) is obtained such that the part of F outside B^2 is not affected by the cut. By a small modification of F, which leaves untouched everything outside B^2, moreover a new normal Dehn surface F_1 is obtained from F of type (p, r) such that :

(i) To the point b correspond two points b_2 and b_3 of F_1,

(ii) $m(b_1) = 1$, $m(b_2)$ and $m(b_3) \geqq 0$, and $m(b_2) + m(b_3) < m(b)$,

(iii) F_1 is obtained from $f(S)$ by a deformation(see [4] for details).

Therefore by performing these methods and referring to [4], in general we may assume that the multiplicity of any branch point of $f(S)$ is always one where $f(S)$ is a singular orientable surface of type (p, r).

Now we shall refer to [5] as follows : For simplicity let N^3 be a simply-connected, closed 3-manifold and $f : D^2 \longrightarrow N^3$ continuous map of a 2-disk such that $f \mid \partial D^2$ is a homeomorphism onto a knot K in N^3. By a modification of f within a regular neighborhood of $f(D^2)$, it may be assumed that the singularities of f consist of a finite number of

(1) Double curves, *i. e.*, pairs of curves on D^2 which are matched by f,

(2) Triple points, at which 3 pairs of double curves cross,

(3) Branch points, common endpoints of one or more pairs of matched double curves.

Furthermore these singularities may be moved slightly by an ambient isotopy of N^3 such that.

(4) No triple point or branch point occurs on ∂D^2,

(5) Only a finite number of double points occurs on ∂D^2.

It may be assumed by the above discussions that the multiplicity of any branch point of $f(D^2)$ is always one. Futhermore it is well-known from [5] that all the triple points can be eliminated from $f(D^2)$. And all the branch points also can be eliminated from $f(D^2)$ with replacing a semidisk having a double curve which is a curve, matched by f, whose two boundary points consist of a branch point and a point of $f(\partial D^2)$, by a new semidisk having no singularity. At last we can obtain such a singular 2-disk which has clasp arcs as singularities.

Now we shall show in § 3 that the following Theorem, which is well-known from [5], is also obtained by a different method which is obtained from Theorem 2. 3.

Theorem 3. 6. *Let N^3 be a simply-connected, closed 3-manifold and K a nontrivial knot in N^3. Then there is a singular 2-disk of K in N^3 that has clasp and ribbon arc singularities.*

And the following Theorem 3. 9 will be proved in the section 3.

Theorem 3. 9. *Let M^3 be a simply-connected 3-manifold with $\partial M^3=S^2$. Suppose that for any finite set of knots $K_i(i=1, 2, \cdots, n)$ in M^3 such that $\partial K_i \subset \partial M^3$, there is a collar neighborhood V^3 of ∂M^3 in M^3 with $K_i \subset V^3$ for $i = 1, 2, \cdots, n$. Then M^3 is diffeomorphic to a 3-disk D^3.*

The auther wishes here to express their sincere graditude to Professor F. Uchida at Yamagata University and O. Saeki at Hiroshima University for their helpful suggestions and kind criticisms.

1. Preliminaries

In this § several Definitions and Theorems will be prepared in order to prove the main Theorem2. 3. And Theorem 1. 6, Lemma 1. 4 and Corollary 1. 5 in this section are referred from [6].

Definition1. 1. Let *M^m, N^n be submanifolds of a p-manifold V^p, which are of dimensions m, n respectively. If for a point $x \in M^m \cap N^n$ the tangent space at x in V^p is spanned by all the vectors of tangent spaces at x in M^m and N^n, then M^m and N^n are said to intersect transversely at x in V^p. If $m+n<p$, assume that*

$$M^m \cap N^n = \phi.$$

Definition1. 2. *Let M^m be a submanifold of V^p and U an open neighborhood of M^m in V^p such that*

$$f : U \longrightarrow M^m \times R^{p-m}$$

is a diffeomorphism with $f(x)=x\times 0$ for any point $x \in M^m$. Then U is called a product neighborhood of M^m in V^p.

Theorem 1. 3 [3]. Let *M^m, N^n be submanifolds of a p-manifold V^p. Suppose that M^m has a product neighborhood in V^p. And let $f : U \longrightarrow M^m \times R^{p-m}$ be a product neighborhood of M^m, then $f^{-1}(M^m\times x)$ and N^n have transverse intersections in V^p for some $x \in R^{p-m}$.*

Now the Sard's Theorem is referred from [6] for a time. Define a subset

$$C^n(x, r)$$

of an n-dimensional Euclidean space R^n as follows :

$$C^n(x, r)= \{y=(y_1, \cdots, y_n) \in R^n ; \mid x_i-y_i \mid < r/2\}$$

where $x=(x_1,\cdots, x_n)$. The volume of $C^n(x, r)$ is r^n. Let Q be a subset of R^n.

Now for any positive number $\varepsilon>0$ assume that there are subsets

$$C^n(x_i, r_i)\,(i=1, 2,\cdots)$$

such that

$$Q \subset \bigcup_{i=1}^{\infty} C^n(x_i, r_i),\ \sum_{i=1}^{\infty} r_i^{\,n} < \varepsilon,$$

then the measure of Q is defined as 0 and then $m(Q)=0$. Let

$$\{Q_i\ ;\ =1, 2,\cdots\}$$

be a family of subsets of R^n with $m(Q_i)=0$ for $i=1, 2, \cdots$, then it follows that

$$m(\cup Q_i)=0.$$

Lemma 1. 4 [6, Lemma 4. 8]. *Let U be an open subset of R^n and $f : U \longrightarrow R^m$ a smooth map. If $m \geqq 2n$, then for any positive number $\varepsilon > 0$ there is a (n, m)-matrix $A = (a_{ij})$ such that $|a_{ij}| < \varepsilon$ and*

$$g : U \longrightarrow R^m$$

is an immersion which is defined by

$$g(x)=f(x)+xA.$$

Now the following Corollary is obtained analogously to Lemma 1. 4.

Corollary 1. 5. *Let U be an open subset of R^n and $f : U \times R \longrightarrow R^m$ a smooth map. If $m \geqq 2n+1$, then for any positive number $\varepsilon > 0$ there is a (n, m)-matrix $A=(a_{ij})$ such that $|a_{ij}| < \varepsilon$ and*

$$g_t : U \longrightarrow R^m$$

is an immersion for any $t \in R$ where the smooth map g_t is defined by

$$g_t(x)=f(x, t)+xA.$$

Theorem 1. 6. *Let M^3 be a connected 3-manifold whose boundary ∂M^3 is a 2-sphere S^2 and let $f, g : I \longrightarrow M^3$ be proper embeddings with $f(i)=g(i) \in \partial M^3$ for $i=0, 1$. Suppose that there is a homotopy $H : I \times R \longrightarrow M^3$ between f and g such that*

$$H(s, t)=f(s) \quad \text{for} \quad t \leqq 1/3$$
$$H(s, t)=g(s) \quad \text{for} \quad 2/3 \leqq t$$

and

$$H(s, t)=f(s)=g(s) \quad \text{for} \quad s \in [0, 1/3] \cup [2/3, 1].$$

Then there exists a regular homotopy $F : I \times R \longrightarrow M^3$ between f and g such that

(1) *F is a δ-approximation of H such that*

$$F(s, t)=f(s)\,(t \leqq 1/3)$$
$$F(s, t)=g(s)\,(t \geqq 2/3)$$

(2) *If the homotopy H is a constant on $U(N)$ for all $t \in R$ where $U(N)$ is an open neighborhood of a closed set $N=[0, 1/3] \cup [2/3,1] \subset I$, then*

$$F(s, t)=f(s)=g(s) \quad \text{for all } s \in N,\ t \in R$$

Proof. Now fix a collar neighborhood V of ∂M^3 in M^3 such that V is diffeomorphic to $S^2 \times [0, 2)$ where assume that $V_1=S^2 \times [0, 1) \subset V \subset M^3$. Then it may be assumed clearly that

$$f(s)=g(s) \in \mathrm{Cl}\, V_1 \qquad \text{for all } s \in [0, 1/3] \cup [2/3, 1]$$

and moreover

$$f(s), g(s) \in M^3 - \mathrm{Cl}\, V \qquad \text{for all } s \in ((1/3)+\varepsilon, (2/3)-\varepsilon)$$

Now by adding $S^2 \times I$ to M^3 such that $S^2 \times \{1\}$ may be identified with ∂M^3 by a suitable diffeomorphism if necessary it may be assumed evidently that there exists a smooth homotopy

$$H : I \times R \longrightarrow M^3$$

between f and g such that

$$H(s, t) = f(s) \qquad \text{for} \qquad t \leqq \varepsilon$$
$$H(s, t) = g(s) \qquad \text{for} \qquad 1 - \varepsilon \leqq t$$

and

$$H(s, t) = f(s) = g(s) \qquad \text{for} \qquad s \in U(N),\ t \in R$$

where $N = I - (1/3, 2/3)$ and

$$U(N) = [0, (1/3) + \varepsilon) \cup ((2/3) - \varepsilon, 1]$$

is an open neighborhood of N in $[0, 1]$ and ε is a sufficiently small positive real number. Then it may be assumed evidently that

$$H(s, t) = f(s) \qquad \text{for} \quad s \in I,\ t \leqq \varepsilon < 1/3$$
$$H(s, t) = g(s) \qquad \text{for} \quad s \in I,\ t \geqq 1 - \varepsilon > 2/3$$
$$H(s, t) = f(s) = g(s) \qquad \text{for} \quad s \in U(N),\ t \in I.$$

Now by considering that on the domain $U(N) \times R$ and $I \times (R - [\varepsilon, 1 - \varepsilon])$, the Jacobi matrix of H

$$J(H \mid (, t))$$

has rank 1 for each fixed t and for any variable s, then let

$$\{(U_i, \varphi_i)\ ;\ i = 1, \cdots, m\}$$

be an open covering of $M^3 - V$ which consists of coordinate neighborhoods of M^3 and such that

$$U_i \cap \partial M^3 = \phi,\ U_i \cap V_i = \phi \qquad \text{for all } i.$$

Now consider the following finite open covering

$$\{W_i \times (t_i - 3\delta_i, t_i + 3\delta_i)\ ;\ = 1, \cdots, n\}$$

of $[(1/3) + \varepsilon, (2/3) - \varepsilon] \times [\varepsilon, 1 - \varepsilon] \subset I \times R$ such that

(1) $\mathrm{Cl}\, W_i \times [t_i - 3\delta_i, t_i + 3\delta_i]$ is compact for each i

(2) (W_i, σ_i) is a coordinate neighborhood for I such that
$\sigma_i(W_i) = \mathrm{Int} D^1(3)$, $\mathrm{Cl}\, W_i \subset (1/3, 2/3)$, $[t_i - 3\sigma_i, t_i + 3\sigma_i] \subset \mathrm{Int}\, I$ for all i

(3) $\{\sigma_i^{-1}(\mathrm{Int} D^1(1)) \times (t_i - \delta_i, t_i + \delta_i)\ ;\ = 1, \cdots, n\}$ is an open covering of
$$[(1/3) + \varepsilon, (2/3) - \varepsilon] \times [\varepsilon, 1 - \varepsilon].$$

Note that it is easy to look out the open covering which satisfies the conditions above :

$$\{W_i \times (t_i - 3\delta_i, t_i + 3\delta_i)\ ;\ = 1, \cdots, n\}$$

for the covering $(U_i, \varphi_i)(i = 1, \cdots, m)$. Because, let

$$(s, t) \in [(1/3) + \varepsilon, (2/3) - \varepsilon] \times [\varepsilon, 1 - \varepsilon]$$

be any point and in order to obtain the desired open covering

$$\{(s - \mu, s + \mu) \times (t - \nu, t + \nu)\}$$

of

$$[(1/3) + \varepsilon, (2/3) - \varepsilon] \times [\varepsilon, 1 - \varepsilon] \subset I \times R,$$

choose positive real numbers μ, $\nu > 0$ so small such that the following conditions are satisfied :

(1) $(s - 3\mu, s + 3\mu) \times (t - 3\nu, t + 3\nu) \subset (1/3, 2/3) \times (0, 1)$.

(2) *Each* $(s - 3\mu, s + 3\mu) \times (t - 3\nu, t + 3\nu)$ has a compact closure in

$$(1/3, 2/3) \times (0, 1).$$

From these discussions above by choosing μ, ν so small and by permitting repititions with respect to the subscripts of U_i, and at last by rewriting all the subscript about $\{U_i\}$, as a result it may be assumed that $n=m$ and that the following condition is satisfied :

$$(*) \qquad H(W_i \times (t_i - 3\delta_i, t_i + 3\delta_i)) \subset U_i \qquad \text{for} \quad i=1, \cdots, n.$$

Now I shall prove by induction for n as follows : First set $F_0 = H$. Then by the inductive assumption suppose that

$$F_{i-1} : I \times R \longrightarrow M^3 \qquad \text{for} \qquad i-1<n$$

is defined already and $J(F_{i-1} \mid I \times \{t\})$ has rank 1 for all the points of

$$N_{i-1} = (K \cup (U(N) \times R) \cup (\bigcup_{k=1}^{i-1} \sigma_k^{-1}(\mathrm{Int}D^1(1)) \times (t_k - \delta_k, t_k + \delta_k)$$

where set $K = I \times (R - [\varepsilon, 1-\varepsilon])$. And moreover it may be assumed that

$$F_{i-1} : I \times R \longrightarrow M^3$$

satisfies the above condition(*).

Let $\Psi : R^n \longrightarrow R$ be a smooth function such that

(1) $\Psi(x)=1$ for $|x| \leqq 1$

(2) $\Psi(x)=0$ for $|x| \geqq 2$

(3) $\Psi(x)=\Psi(x')$ for $|x| = |x'|$

(4) $0<\Psi(x)<1$ for $1<|x|<2$.

And in the case $n=1$, define a smooth function

$$\Psi_i : I \longrightarrow R$$

by

$$\Psi_i(s) = \Psi(\sigma_i(s)) \qquad \text{for} \quad s \in W_i$$
$$\Psi_i(s) = 0 \qquad \text{for} \quad s \in I - W_i.$$

And let

$$\Phi_i : I \times R \longrightarrow R$$

be a smooth function which is defined by

$$\Phi_i(s, t) = \Psi_i(s) \circ \Psi((t-t_i)/\delta_i) \qquad \text{for all } s \in I \text{ and all } t \in R.$$

Now define the following smooth function :

$$G : \mathrm{Int}D^1(3) \times (t_i - 3\delta_i, t_i + 3\delta_i) \longrightarrow R^3$$

by

$$G(\tau, t) = F_i(\tau, t) + \Phi_i(s, t) \cdot (\tau A)$$

where A is a matrix of type (1, 3) and set $\tau = \sigma_i(s)$.

Since all the entries of A are all sufficiently near to 0, therefore

$$J(G \mid \mathrm{Int}D^1(2) \times \{t\})$$

has rank 1 at all the points :

$$(s, t) \in N_{i-1} \cap (\sigma_i^{-1}(\mathrm{Int}D^1(2)) \times (t_i - 2\delta_i, t_i + 2\delta_i)).$$

And moreover it may be assumed evidently that for any i

$$G(\mathrm{Int}D^1(3) \times (t_i - 3\delta_i, t_i + 3\delta_i)) \subset \varphi_i(U_i).$$

Now define a new smooth function

$$F_i : I \times R \longrightarrow M^3$$

by

$$F_i(s,\ t)=F_{i-1}(s,\ t) \quad \text{for} \quad (s,\ t)\notin(\sigma_i^{-1}(D^1(2))\times[t_i-2\delta_i,\ t_i+2\delta_i])$$
$$F_i(s,\ t)=\varphi_i^{-1}\circ G(\sigma_i(s),\ t) \quad \text{for} \quad (s,\ t)\in(\sigma_i^{-1}(D^1(2))\times[t_i-2\delta_i,\ t_i+2\delta_i]).$$

Now let

$$N_i=N_{i-1}\cup(\sigma_i^{-1}(D^1(1))\times[t_i-\delta_i,\ t_i+\delta_i])$$
$$=(K\cup(U(N)\times R)\cup(\bigcup_{j=1}^{i}\sigma_j^{-1}(D^1(1))\times[t_j-\delta_j,\ t_j+\delta_j])$$

where $K=I\times(R-[\varepsilon,\ 1-\varepsilon])$.

Then from the discussions above it follows clearly that

$$J(F_i \mid (\sigma_i^{-1}(D^1(1))\times\{t\})$$

has rank 1 for all $t\in[t_i-\delta_i,\ t_i+\delta_i]$ and also at all the points $(s,\ t)\in N_i$, *i. e.*,

$$J(F_i \mid (\ ,\{t\}))$$

has rank one. And by choosing all the entries of A so small it may be assumed actually that

$$F_i : I\times R \longrightarrow M^3$$

satisfies the condition (*). And define a smooth function

$$F : I\times R \longrightarrow M^3$$

which is the desired regular homotopy beween f and g, by

$$F(s,\ t)=F_n(s,\ t).$$

This completes the proof of Theorem1.6.

2. Proof of the Main Theorem 2. 3

Definition 2. 1. *Let M^n and V^{2n} be an n-manifold and a 2n-manifold respectively and*

$$f : M^n \longrightarrow V^{2n}$$

a smooth map. Suppose that $\{f^{-1}(q)\} = \{p_1, p_2\}$ consists of two different points for a point $q\in V^{2n}$ and that

$$T_{p_1}f(T_{p_1}(M^n))\oplus T_{p_2}f(T_{p_2}(M^n))=T_p(V^{2n}).$$

Then f is said to intersect transversely at q.

Definition 2. 2. *Let $f : M^n \longrightarrow V^{2n}$ be an immersion under the notations of the above Definition* 2. 1. *Suppose the following conditions for any point $q\in f(M^n)$.*

(i) *$f^{-1}(q)$ consists of only one point,*

(ii) *f intersects transversely at q.*

If either (i) *or* (ii) *is satisfied for f and any point $q\in f(M^n)$, then f is called a complete immersion.*

Now let X, Y be metric spaces and $\delta : X \longrightarrow (0,\infty)$ a positive real-valued continuous function. For two continuous functions $f,\ g : X \longrightarrow Y$,

$$\rho(f(x),\ g(x))<\delta(x)$$

is satisfied for any $x\in X$ where ρ is a metric of Y, then g is said a δ-approximation of f.

Theorem 2.3. *Let* $\delta : M^3 \times R \longrightarrow (0, \infty)$ *be a positive real-valued continuous function. Under the hypotheses of Theorem* 1. 6, *moreover suppose that*

$$f, g : I \longrightarrow M^3$$

are proper embeddings with $f(i) = (i) \in \partial M^3 = S^2$ *for* $i = 0, 1$. *Then for a regular homotopy* H *between* f *and* g *in Theorem* 1. 6, *there is a complete immersion*

$$G : I \times R \longrightarrow M^3 \times R$$

such that

(1) *G is level-preserving,*

(2) *G is a δ-approximation of $H \times i$,*

(3) *G is homotopic to H* rel. $I \times (R - \text{Int } I) \cup N \times R$:

$$G \mid (I \times (R - \text{Int } I)) \cup N \times R = H \times i \mid (I \times (R - \text{Int } I)) \cup N \times R$$

where $N = [0, 1/3] \cup [2/3, 1] \subset I$ *and i is an identity map,*

(4) $G : I \times R \longrightarrow M^3 \times R$ *is a regular homotopy between f and g.*

Proof. Now use the same notations as in Theorem1.6 and its Proof. From Theorem1. 6 for a homotopy H, there is a regular homotopy

$$F : I \times R \longrightarrow M^3 \times R$$

such that F is a homotopy between f and g and F satisfies the following conditions :

$$F(s, t) = (f(s), t) \qquad \text{for } s \in I,\ t \leqq \varepsilon < 1/3$$
$$F(s, t) = (g(s), t) \qquad \text{for } s \in I,\ t \geqq 1 - \varepsilon > 2/3$$
$$F(s, t) = f(s) = g(s) \qquad \text{for } s \in U(N),\ t \in R$$

where $N = [0, 1/3] \cup [2/3, 1]$, and let

$$U(N) = [0, (1/3) + \varepsilon) \cup ((2/3) - \varepsilon, 1].$$

Now by considering that the restricted map of F to the domain

$$U(N) \times R \text{ and } I \times (R - [\varepsilon, 1 - \varepsilon])$$

is a complete immersion, *i. e.*, it is an embedding in this case, choose an open covering

$$(U_i, \varphi_i)\ (i = 1, 2, \cdots, n)$$

of $M^3 - V$ such that

$$U_i \cap \partial M^3 = \phi,\ U_i \cap V_1 = \phi\ (i = 1, 2, \cdots, n).$$

Then by using the same methods as the proof of Theorem1.6 there is an open covering :

$$\{W_i \times (t_i - 3\delta_i, t_i + 3\delta_i)\ ;\ i = 1, \cdots, n\}$$

of $[(1/3) + \varepsilon, (2/3) - \varepsilon] \times [\varepsilon, 1 - \varepsilon]$ in $I \times I$ such that the following conditions are satisfied :

(a). $\text{Cl}\, W_i \times [t_i - 3\delta_i, t_i + 3\delta_i]$ is compact for each i

(b). Each (W_i, σ_i) is a coordinate neighborhood of I such that : for all i

$$\sigma_i(W_i) = \text{Int} D^1(3),\ \text{Cl}\, W_i \subset (1/3, 2/3),\ (t_i - 3\delta_i, t_i + 3\delta_i) \subset \text{Int } I$$

(c). $\{\sigma_i^{-1}(\text{Int} D^1(1) \times (t_i - \delta_i, t_i + \delta_i);\ i = 1, 2, \cdots, n\}$ is an open covering of

$$[(1/3) + \varepsilon, (2/3) + \varepsilon] \times [\varepsilon, 1 - \varepsilon] \text{ in } I \times I.$$

(d). $f, g : I \longrightarrow M^3$ are proper embeddings and

$$F \mid W_i \times (t_i - 3\delta_i, t_i + 3\delta_i)$$

is an embedding such that

$$F(W_i \times (t_i - 3\delta_i, t_i + 3\delta_i)) \subset U_i \times R \qquad \text{for } i = 1, \cdots, n$$

(e). Since F is a regular homotopy, by choosing suitable local coordinates(U_i, φ_i) $(i = 1, 2, \cdots, n)$, the smooth each map

$$(\varphi_i \times i) \circ F : W_i \times (t_i - 3\delta_i, t_i + 3\delta_i) \longrightarrow \mathrm{Int} D^3 \times R$$

satisfies that, for all i

$$((\varphi_i \times i) \circ F)(W_i \times (t_i - 3\delta_i, t_i + 3\delta_i)) \subset (\mathrm{Int} D^3 \times R) \cap (0 \times 0 \times R \times R).$$

Note that "use the Inverse Function Theorem to show the condition(e) above".

Now I shall prove by the induction for n, that is, the smooth map

$$G_i : I \times R \longrightarrow M^3 \times R \qquad \text{for each } i \leqq n$$

will be constructed and subsets $N_i \subset I \times R$ also will be constructed such that

$$G_i \mid N_i : N_i \longrightarrow M^3 \times R$$

is a complete immersion. And modify the open covering

$$\{W_i \times (t_i - 3\delta_i, t_i + 3\delta_i) \; ; \; i = 1, \cdots, n\}$$

of

$$[(1/3) + \varepsilon, (2/3) - \varepsilon] \times [\varepsilon, 1 - \varepsilon] \text{ in } I \times I$$

satisfies the conditions above (a), (b), (c), (d), and (e) for each i.

At first set

$$G_0 = F.$$

And by the inductive assumption, suppose that there is a smooth map

$$G_{i-1} : I \times R \longrightarrow M^3 \times R$$

where

$$G_{i-1} \mid I \times (R - \mathrm{Int}\, I) \cup N \times R = G_0 \mid I \times (R - \mathrm{Int}\, I) \cup N \times R.$$

And also by the inductive assumption suppose evidently that the open covering of

$$[(1/3) + \varepsilon, (2/3) - \varepsilon] \times [\varepsilon, 1 - \varepsilon] \text{ in } I \times I :$$

$$\{W_i \times (t_i - 3\delta_i, t_i + 3\delta_i) \; ; \; i = 1, \cdots, n\}$$

and the smooth map

$$G_{i-1} : I \times R \longrightarrow M^3 \times R$$

satisfies the conditions above (a), (b), (c), (d), and(e).

Now set

$$K = I \times (R - (\varepsilon, 1 - \varepsilon))$$

and let

$$N_{i-1} = (K \cup (N \times R) \cup (\bigcup_{j=1}^{i-1} (\sigma_j^{-1}(D^1(1)) \times [t_j - \delta_j, t_j + \delta_j]).$$

Then since by the inductive assumption $G_{i-1} \mid N_{i-1}$ is a complete immersion for $i - 1 < n$, and then

$$(G_{i-1} \mid N_{i-1})^{-1}(q)$$

consists of two points at most for any $q \in G_{i-1}(N_{i-1})$ and if

$$(G_{i-1} \mid N_{i-1})^{-1}(q)$$

consists of just two points, then $G_{i-1} \mid N_{i-1}$ has a transverse intersection at q, that is,

$$G_{i-1} \mid N_{i-1} : N_{i-1} \longrightarrow M^3 \times R$$

is a complete immersion.

Now consider the following map

$$(\varphi_i \times i) \circ G_{i-1} \circ (\sigma_i^{-1} \times i) : \mathrm{Int}\, D^1(3) \times (t_i - 3\delta_i, t_i + 3\delta_i) \longrightarrow \mathrm{Int}\, D^3 \times R$$

and a projection

$$\pi : R^4 \longrightarrow R^2$$

such that

$$\pi(x_1, x_2, x_3, x_4) = (x_1, x_2).$$

Then from the condition (e) above it follows that

$$(\varphi_i \times i) \circ G_{i-1} \circ (\sigma_i^{-1} \times i)(\text{Int } D^1(3) \times R) \subset \pi^{-1}(0, 0).$$

Now consider also the following smooth map

$$\pi \circ (\varphi_i \times i) \circ G_{i-1} : N_{i-1} \cap (G_{i-1})^{-1}(U_i \times R) \longrightarrow R^2$$

where let

$$N_{i-1} = (K_1 \cup (U(N) \times R) \cup (\bigcup_{j=1}^{i-1} (\sigma_j^{-1}(D^1(3)) \times (t_j - 3\delta_j, t_j + 3\delta_j)))$$

and let

$$K_1 = I \times (R - [\varepsilon, 1-\varepsilon]).$$

Then by using the Sard's Theorem the measure of all the critical values of

$$\pi \circ (\varphi_i \times i) \circ G_{i-1} \mid N_{i-1} \cap (G_{i-1})^{-1}(U_i \times R)$$

is equal to 0. Therefore by the inductive assumption,these conditions above are satisfied for the smooth map

$$G_{i-1} \mid N_{i-1} : N_{i-1} \longrightarrow M^3 \times R$$

and for the covering, which covers N_{i-1}, and consists of the open subsets :

$$\{W_k \times (t_k - 3\delta_k, t_k + 3\delta_k) \ ; \ k = 1, 2, \cdots, i-1\}.$$

Now by using the Sard's Theorem choose $c_i \in R^2$ such that the following conditions are satisfied :

(i) c_i is sufficiently near to (0, 0)

(ii) c_i is a regular value of the smooth map

$$\pi \circ (\varphi_i \times i) \circ G_{i-1} \mid N_{i-1} \cap G_{i-1}^{-1}(U_i \times R) : N_{i-1} \cap G_{i-1}^{-1}(U_i \times R) \longrightarrow R^2$$

(iii) If $p_1, p_2 \in N_{i-1} \cap (I \times I)$ and

$$G_{i-1}(p_1) = G_{i-1}(p_2) \in U_i \times (t_i - \delta_i, t_i + \delta_i).$$

Then it is satisfied that

$$(\varphi_i \times i) \circ G_{i-1}(p_1) = (\varphi_i \times i) \circ G_{i-1}(p_2) \notin \pi^{-1}(c_i).$$

Now note that since $G_{i-1} \mid N_{i-1}$ is a complete immersion,the subset of $M^3 \times R$ at which $G_{i-1} \mid N_{i-1}$ intersects transversely is a finite set. It will be shown later in Theorem2.6 that any point at which $G_{i-1} \mid N_{i-1}$ intersects transversely is an isolated point in $M^3 \times R$. And note that it may be assumed if necessary that

$$(G_{i-1} \mid N_{i-1})(\rho_i(D^1(2)) \times [t_i - 2\delta_i, t_i + 2\delta_i] - \rho_i(\text{Int}D^1(1)) \times (t_i - \delta_i, t_i + \delta_i))$$

has no transverse point for $G_{i-1} \mid N_{i-1} : N_{i-1} \longrightarrow M^3 \times R$.

Now define a smooth map $\Psi : R \longrightarrow R$ such that the following conditions are satisfied :

(1) $\Psi(x) = 1$ for $|x| \leqq 1$

(2) $\Psi(x) = 0$ for $|x| \geqq 2$

(3) $0 < \Psi(x) < 1$ for $1 < |x| < 2$.

Then by using the map Ψ define the following smooth map

$$\Psi_i : I \longrightarrow R$$

by

$$\Psi_i(s) = \Psi(\sigma_i(s)) \qquad (s \in W_i)$$

$\Psi_i(s)=0 \qquad (s\in I-W_i)$

And moreover define a smooth map

$$\Phi_i : I\times R \longrightarrow R$$

by

$$\Phi_i(s,\ t)=\Psi_i(s)\cdot\Psi((t-t_i)/\delta_i) \text{ for all } (s,\ t)\in I\times R.$$

Now define a new smooth map

$$G_i : I\times R \longrightarrow M^3\times R$$

by

$$G_i(s,\ t)=G_{i-1}(s,\ t) \textit{ for } (s,\ t)\in I\times R-\sigma_i^{-1}(D^1(2))\times[t_i-2\delta_i,\ t_i+2\delta_i]$$
$$G_i(s,\ t)=(\varphi_i\times i)^{-1}((\varphi_i\times i)\circ G_{i-1}(s,\ t)+c_i\Psi(\sigma_i(s))\cdot\Psi((t-t_i)/\delta_i))$$
$$=(\varphi_i\times i)^{-1}((\varphi_i\times i)\circ G_{i-1}(s,\ t)+c_i\Phi_i(s,\ t)) \text{ for } (s,\ t)\in W_i\times[t_i-2\delta_i,\ t_i+2\delta_i].$$

Now it follows evidently from the above conditions (i),(ii),(iii) that

$$G_i : I\times R \longrightarrow M^3\times R$$

is an immersion. And set

$$N_i=N_{i-1}\cup\sigma_i^{-1}(D^1(1))\times[t_i-\delta_i,\ t_i+\delta_i].$$

Then since from the definition of N_i,

$$N_i\cap([0,\ 1]\times[0,\ 1])$$

is compact,therefore it is evident that

$$(G_i \mid N_i)^{-1}(q)$$

consists of two points at most for any $q\in G_i(N_i)$ and

$$G_i \mid N_i : N_i \longrightarrow M^3\times R$$

is a complete immersion. Because set

$$G_i(s,\ t)=(g_t(s),\ t),$$

and assume that

$$G_i(s,\ t)=(g_t(s),\ t)\in U_j\times R.$$

Then set

$$(\varphi_j\times i)\circ G_i(s,\ t)=(x_1(s,\ t),\ x_2(s,\ t),\ x_3(s,\ t),\ t)$$

$$(\star)\qquad \begin{bmatrix} \dfrac{\partial x_1(s,t)}{\partial s} & \dfrac{\partial x_1(s,t)}{\partial t} \\ \dfrac{\partial x_2(s,t)}{\partial s} & \dfrac{\partial x_2(s,t)}{\partial t} \\ \dfrac{\partial x_3(s,t)}{\partial s} & \dfrac{\partial x_3(s,t)}{\partial t} \\ 0 & 1 \end{bmatrix}$$

Now since $G_i \mid N_i : N_i \longrightarrow M^3\times R$ is an immersion, therefore the above (4, 2)-matrix (☆), which is obtained from $(\varphi_j\times i)\text{o}G_i(s,\ t)$, has rank 2. If

$$(s,\ t)\in\sigma_i^{-1}(D^1(1))\times[t_i-\delta_i,\ t_i+\delta_i],$$

then from the condition (e) it follows that

$$(\varphi_j\times i)\circ G_i(s,\ t)=(0,\ 0,\ x_3(s,\ t),\ t).$$

On (☆) above, it is evident generally for some i that

$$\frac{\partial x_i(s, t)}{\partial s} \neq 0.$$

Now as before assume that

$$G_i : I \times R \longrightarrow M^3 \times R$$

is a complete immersion which satisfies the conditions (a), (b), (c), (d), and (e).

Now from the discussions above define a map

$$G : I \times R \longrightarrow M^3 \times R$$

by

$$G(s, t) = G_n(s, t).$$

This completes the proof of Theorem2. 3.

Now let M^3 be a 3-manifold whose boundary is a 2-sphere S^2 and

$$f, g : I \longrightarrow M^3$$

proper embeddings such that $f(i) = g(i)$ $(i = 0, 1)$ and f is homotopic to g rel. ∂I.

Now from Theorem2. 3 there is a regular homotopy

$$G : I \times R \longrightarrow M^3 \times R$$

between f and g, and moreover it may be assumed that

$$G : I \times R \longrightarrow M^3 \times R$$

is a complete immersion. Now set

$$G(s, t) = (g_t(s), t).$$

Let $(q, t) \in M^3 \times R$ be such a point that G has a transverse intersectin at (q, t). Now assume that

$$G(s_1, t) = (g_t(s_1, t), t) = G(s_2, t) = (g_t(s_2), t) = (q, t) \quad (s_1 \neq s_2).$$

Then by seeing from the proof of Theorem2. 3, and by choosing a suitable local neighborhood

$$(U_i \times R, \varphi_i \times i)$$

of (q, t) in $M^3 \times R$ with respect to (s_1, t), set

$$(\varphi_i \times i) \circ G(s, t) = (x_1(s, t), x_2(s, t), x_3(s, t), t).$$

Then on a suitable neighborhood of (s_1, t) from the condition (e) it follows that

$$x_1(s, t) = x_2(s, t) = 0.$$

That is, it follows that on a suitable neighborhood of (s_1, t)

$$(\varphi_i \times i) \circ G(s, t) = (0, 0, x_3(s, t), t).$$

Since moreover by the assumption of the Transversality at (q, t) the Jacobi matrix of $(x_1(s, t), x_2(s, t))$ at (s_2, t) has rank 2, therefore on considering a projection

$$\pi : R^4 \longrightarrow R^3$$

which is defined by

$$\pi(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3),$$

then by using the Inverse Function Theorem for a suitable neighborhood of the point $(x_1(s, t), x_2(s, t))$ of $\pi \circ (\varphi_i \times i) \circ G(s_2, t)$ it follows that

$$(\bigstar) \quad \begin{cases} h(t) = \pi \circ (\varphi_i \times i) \circ G(s, t) = (0, 0, x_3(s, t)) \subset R^3 \text{ for } \forall (s, t) \in W_1, \\ \\ h(t) = \pi \circ (\varphi_i \times i) \circ G(s, t) = (x_1(s, t), x_2(s, t), 0) \text{ for } \forall (s, t) \in W_2, \end{cases}$$

where each W_i is a suitable neighborhood of (s_i, t) for each i.

Because since $(x_1(s, t), x_2(s, t))$ has rank 2, by the Inverse Function Theorem there is an inverse function ρ of $(x_1(s, t), x_2(s, t))$ on a suitable neighborhood V of the point $(x_1(s_2, t), x_2(s_2, t))$. Let $p : R^4 \longrightarrow R^2$ be a projection defined by

$$p(x_1, x_2, x_3, x_4) = (x_1, x_2).$$

Then evidently it follows that

$$\rho \circ p \circ (\varphi_i \times i) \circ G(s, t) = (s, t)$$
$$p \circ (\varphi_i \times i) \circ G \circ \rho(x_1, x_2) = (x_1, x_2).$$

Therefore on a suitable neighborhood of (s_2, t),the surface

$$(\varphi_i \times i) \circ G(s, t) = (x_1(s, t), x_2(s, t), x_3(s, t), t).$$

is represented as follows ;

$$(\varphi_i \times i) \circ G(s, t) = (x_1(s, t), x_2(s, t), x_3(x_1(s, t), x_2(s, t)), t).$$

Now by using the following coordinate transformation :

$$(x_1, x_2, x_3, t) \longrightarrow (x_1, x_2, x_3 - x_3(x_1, x_2), t),$$

the condition (★) is satisfied for the Invariance of the Transversality for a diffeomorphism. Therefore it may be assumed that the condition (★) is always satisfied at any point at which the map

$$G : I \times R \longrightarrow M^3 \times R$$

has transverse intersections.

Definition 2. 4. *Let M^3 be a 3-manifold whose boundary is a 2-sphere S^2 and*

$$f, g : I \longrightarrow M^3$$

proper embeddings such that $f(i) = g(i)$ $(i=0, 1)$ and f is smoothly homotopic to g rel. *$(f(\partial I) = g(\partial I))$. Then suppose that there is a smoothly homotopy*

$$f_t : I \longrightarrow M^3$$

between f and g such that the following conditions are satisfied :

(1). *$f_t = f$ for $t \leqq 0$, $f_t = g$ for $t \geqq 1$, and*

$$f_t(i) = f(i) = g(i) \quad \text{for all } t \in R,\ i=0, 1.$$

(2). *If $f_t(s) = f_t(s')$ at which f_t has transverse self-intersection points which means that the two tangent vectors of the curves $f_t(I)$ at a point $f_t(s) = f_t(s')$ span a 2-dimensional vector subspace, then from the discussions above it follows that $f_t(s')$ is sationary with respect to the x_3-axis, that is, the x_3-component of $f_t(s')$ is 0, and $f_t(s)$ moves with respect to the x_3-axis, that is, the x_1, x_2-componenets of are 0, in a suitable neighborhood of t, therefore*

$$f_t(s') \text{ and } f_t(s)$$

have a transverse self-intersection point, which means as above, with respect to the variable t. And f_t has finite intersection points as above for $t_1, \cdots, t_n$ where $t_i \in \mathrm{Int}\, I$ for $i=1, \cdots, n$.

(3). *$f_t : I \longrightarrow M^3$ is a proper embedding for any $t \in R - \{t_i ; i=1, 2, \cdots, n\}$.*

(4). *If set $G(s, t) = (f_t(s), t)$, then*

$$G : I \times R \longrightarrow M^3 \times R$$

is a complete immersion.

Then $f_t : I \longrightarrow M^3$ is said a completely regular homotopy between f and g.

Now let M^3 be a 3-manifold whose boundary is a 2-sphere S^2 and

$$f,\ g : I \longrightarrow M^3$$

proper embeddings such that

$$f(i) = g(i)\,(i=0,\ 1)$$

and f is smoothly homotopic to g rel. $(f(\partial I) = g(\partial I))$. Then using Theorem2. 3 there is a complete immersion such that by setting

$$G(s,\ t) = (f_t(s),\ t),$$

it follows that

$$f_t = f\,(t \leqq 0),\ f_t = g\,(t \geqq 1)$$

and morover

$$f_t(i) = f(i) = g(i) \qquad \text{for } i=0,\ 1.$$

Now from Theorem2.3 the following Corollary is obtained :

Corollary 2. 5. *Let M^3 be a connected 3-manifold whose boundary is S^2 and*

$$f,\ g : I \longrightarrow M^3$$

proper embeddings such that

$$f(i) = g(i)\,(i=0,\ 1)$$

and f is smoothly homotopic to g rel. $(f(\partial I) = g(\partial I))$. *Then there is a completely regular homotopy*

$$f_t : I \longrightarrow M^3$$

between f and g.

Now I shall refer to the following Theorem :

Theorem 2. 6 [2]. *Let $G : I \times R \longrightarrow M^3 \times R$ be a complete immersion in Theorem2. 3. Then self-intersectin points are isolated points in $M^3 \times R$.*

3. Singular 2-Disks of Knots

In this § I refer to the Encyclopedic Dictionary of Mathematics,Second Edition by the Mathematical Society of Japan for notations and concepts of isotopies.

Definition 3. 1. *Let $f,\ g : M \longrightarrow N$ be embeddings where M, N are manifolds of dimension m, n, respectively. Suppose that there is an embedding*

$$F : M \times R \longrightarrow N \times R$$

such that the following conditions are satisfied :

(1) *$F(M \times \{t\}) \subset N \times \{t\}$ for any $t \in R$.*

In this case it is said that F is level-preserving, i. e., it preserves the coordinate t.

(2) *$f = f_t$ for $t \leqq 0$ and $g = f_t$ for $t \geqq 1$, by setting*

$$F(p,\ t) = (f_t(p),\ t) \text{ for } p \in M,\ t \in R.$$

Then f and g are said isotopic and F is called an isotopy from f to g. The support Supp F $\subset M$ *of an isotopy F is defined as the closure of*

$$\{p \in M \ ; \ F(p, t) \neq F(p, t') \text{ for some } t, t' \in R\}.$$

Definition 3. 2. *An isotopy $F : M \times R \longrightarrow M \times R$ is called a diffeotopy or an ambient isotopy if the following conditions are satisfied:*

(1) *f_t is the identity map of M for $t \leqq 0$*

(2) *$f_1 = f_t$ for $t \geqq 1$.*

In this § let M^3 be a simply-connected, closed 3-manifold.

Definition 3. 3. Let M^3 *be a simply-connected, closed* 3*-manifold and*

$$f : S^1 \longrightarrow M^3$$

an embedding. Then $f(S^1) = K$ is said a knot in M^3.

Definition 3. 4. *Let M^3 be a simply-connected, closed* 3*-manifold and K a knot in M^3. If there is a* 2*-disk D^2 in M^3 such that $\partial D^2 = K$ and* Int $D^2 \cap K = \phi$, *then K is said trivial. If for a knot K there is no* 2*-disk D^2 such that $\partial D^2 = K$ and* Int $D^2 \cap K = \phi$, *then K is said nontrivial.*

Definition 3. 5. *Let K be a knot in the* 3*-manifold M^3. Suppose that there is a transverse immersion $f : D^2 \longrightarrow M^3$ such that the following conditions are satisfied:*

(1) *$f | \partial D^2 : \partial D^2 \longrightarrow K$ is a diffeomorphism,*

(2) *$\{f^{-1}(q)\}$ consists of at most two points for each $q \in M^3$.*

If $\{f^{-1}(q)\}$ consists of just two points in D^2, then f has a transverse intersection at q and q is said a singular point of $f(D^2)$,

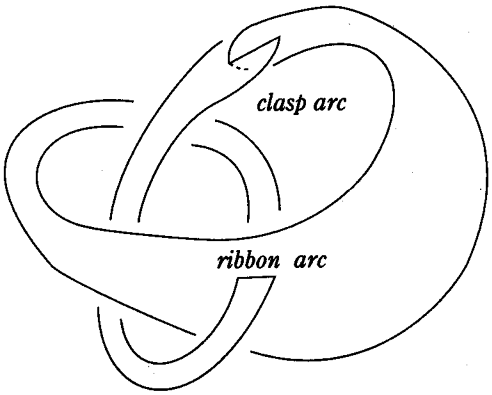
(3) *Each component of singular points of $f(D^2)$ consists of an arc. And these arcs consist of two kinds of arcs as illustrated in Figure* 3. 1, *that is, a clasp arc and a ribbon arc, let l be a singular arc contained into $f(D^2)$. If $\{f^{-1}(l)\}$ consists of such two arcs that one of two bounary points of each arc is contained into ∂D^2 and all other points, in each arc, which contain another boundary point of each arc lies in* Int D^2, *then l is called a clasp arc. If one of $\{f^{-1}(l)\}$ is contained into* Int D^2 *and the two boundary points of another arc lie on ∂D^2, and moreover the interior of this another arc is contained into* Int D^2, *then l is called a ribbon arc. And then the transversely immersed* 2*-disk $f(D^2)$ as above is called a singular* 2*-disk of a knot K and use $\Delta^2(K)$ for the notation instead of $f(D^2)$, that is, set*

$$f(D^2) = \Delta^2(K).$$

Now it is well-known from [5] that for any knot K which homotopic 0 in an orientable closed 3-manifold W^3, there exists a singular 2-disk

$$f : D^2 \longrightarrow W^3$$

having *double curves, tripple points,* and *branch points* such that no triple point or branch point occurs on ∂D^2 and only a finite number of double points occurs on ∂D^2. And then by using the methods in [4] and [5] it is proved that all the triple points and branch points can be eliminated from the singular 2-disk $f(D^2)$ above, as a result, we obtain that the singular 2-disk $f(D^2)$ has at most clasp arcs as singularities, where $f(D^2)$ is called a clasp 2-disk.



$\Delta^2(K)$

Figure 3.1

Now we shall obtain such a singular 2-disk which has at most clasp and ribbon arcs as singularities, by a different method, as follows:

Theorem 3. 6. *There is a singular 2-disk for any nontrivial knot K in a simply-connected, closed 3-manifold M^3 where singularities on the singular 2-disk consists of at most clasp arcs and ribbon arcs.*

Proof. Now set $N^3=M^3-\text{Int}\, D^3$ and $T=K-\text{Int}\, D^3$ where D^3 is a sufficiently small 3-disk in M^3 such that $K\cap D^3$ is trivial. And T is said a knot having ∂ in N^3 for a time. Since N^3 is a simply-connected 3-manifold whose boundary is diffeomorphic to a 2-sphere S^2 and let L be a trivial knot in N^3 with

$$\partial T=\partial L,$$

therefore by using Corollary 2.5 for T and L, there is a completely regular homotopy

$$f_t : I \longrightarrow M^3$$

such that

$$f_0(I)=T,\ f_1(I)=L,\ f_t(\partial I)=f_0(\partial I)\,(\forall t).$$

Now assume that f_t has a self-intersection point for each $t_i(i=1, 2,\cdots, n)$ and for simplicity assume that

$$0<t_1< \cdots <t_n<1.$$

And each f_t is a proper embedding for all

$$t\in I-\{t_i\ ;\ i=1, 2, \cdots, n\}.$$

Since f_t is an isotopy on each open interval of

$$\{(0, t_1), (t_1, t_2), \cdots, (t_{n-1}, t_n), (t_n, 1)\},$$

for each interval there is an ambient isotopy which encloses f_t. Now let

$$0<t_1<s_1<t_2<s_2<t_3< \cdots <t_{n-1}<s_{n-1}<t_n<s_n<1.$$

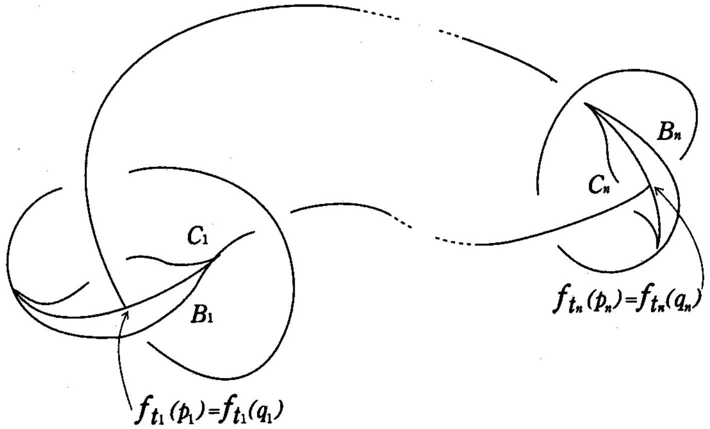


Figure 3.2

Now set $f_{s_i}(I)=T_i(i=1, 2, \cdots, n)$. Then by using suitable ambient isotopies enclosing f_t $(0<t<t_1)$ and $(t_1<t<t_2)$ between knots T and T_1, it follows clearly that the knot T_1 is obtained after having intersected T at some point $f_{t_1}(p_1)=f_{t_1}(q_1)(p_1\neq q_1)$ once itself such that the two tangent vectors at $f_{t_1}(p_1)$ span a 2-dimensional vector subspace. That is, use $f_t(I)$ instead of T if necessary where $t<t_1$ and t is sufficiently enough near to t_1. Therefore seeing from these discussions T is obtained from T_1 after having intersected T_1 only one time itself such that the two tangent vectors at a point of T_1 span a 2-dimensional vector subspace. Then write for the brevity of simplicity for notations as follows:

$$T=(T_1-C_1)\cup B_1.$$

where let $D_1{}^2$ be a 2-cell in Int N^3, then

$$\partial D_1{}^2=C_1\cup B_1,\ C_1\cap B_1=\partial C_1=\partial B_1$$

are satisfied (see Figure 3.2).

Now I shall prove by the induction for n which is the number that

$$f_t(I)(\forall t\in R)$$

has self-intersection points. Now first assume that $f_t(I)(\forall t\in R)$ or $h_t(S^1)$ has only one self-intersection point where $h_t(S^1)$ is a natural extension of $f_t(I)$, for example, $f_{t_1}(I)$, hence $h_{t_1}(S^1)$ has a self-intersection point and each $f_t(I)$ is a knot in N^3 for $t\neq t_1$. Then if necessary, by using a suitable Isotopy Theorem for $f_t : I \longrightarrow N^3$ or $h_t : S^1 \longrightarrow M^3$ with fixing $f_t(\partial I)(\forall t)$ or fixing all the points of D^3 in M^3, it is evident from Figure 3.1 and Figure 3.3 that there is an embedding

$$g_1 : B^2 \longrightarrow M^3$$

and by using a suitable Transversality Theorem for Int B^2 and Int l with fixing all the points of B^2 and the boundary ∂l there is also a transverse self-intersection map

$$g : D^2 \longrightarrow M^3$$

such that $B^2\subset D^2$, g is an extension of g_1 and

$$g|\partial D^2 : \partial D^2 \longrightarrow M^3$$

is an embedding with $g(\partial D^2)=h_0(S^1)$. Therefore it is evident from seeing Figure 3.3 that

$$g(D^2)=\Delta^2(h_0(S^1))\subset M^3$$

is a singular 2-disk having a clasp arc and several ribbon arcs.

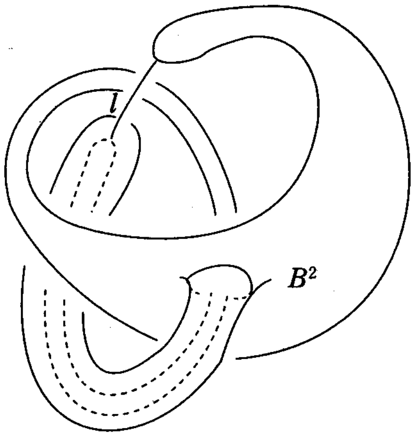


Figure 3.3

Now by the inductive assumption suppose that Theorem 3.6 is true if a knot becomes trivial after having intersected $(n-1)$-times itself such that the two tangent vectors at each intersection point span a 2-dimensional vector subspace in the 3-dimensional tangent space at the intersection point in N^3.

Now suppose that a knot T becomes trivial after having intersected n-times itself such that the two tangent vectors at each intersection point span a 2-dimensional vector subspace in the 3-dimensional tangent space at the intersection point in N^3. Then from the discussions above T_1 is obtained from a knot T after having intersected only one time itself as above, therefore T_1 becomes trivial after having intersected $(n-1)$-times itself. Therefore since there is a singular 2-disk

$$\Delta^2(T_1 \cup (\partial K \cap D^3)$$

for a knot $T_1 \cup (\partial K \cap D^3)$ by the assumption. Therefore by using the Transversality Theorem as above it follows evidently that there is a singular 2-disk $\Delta^2(K)$ which has clasp and ribbon arcs as singularities. This completes the proof of Theorem 3.6.

Definition 3.7. *Let M^3 be a connected, 3-manifold having boundaries. And let D^n be a unit n-disk in a Euclidean space R^n and $f^i : \partial D^i \times D^{3-i} \longrightarrow \partial M^3$ an embedding for $i=0$ 1, 2, 3. Then the quotient space $M^3 \cup h^i$, which is obtained from*

$$M^3 \cup (D^i \times D^{3-i})$$

by identifying with f^i, is called a 3-manifold with a handle attached by f^i where

$$h^i : D^i \times D^{3-i} \longrightarrow M^3 \cup h^i$$

is a natural extention of f^i. And the 3-manifold $M^3 \cup h^i$ is said to be obtained from M^3 by attaching a handle of index i by using an embedding

$$h^i | \partial D^i \times D^{3-i} : \partial D^i \times D^{3-i} \longrightarrow \partial M^3.$$

Now suppose that V^3 is diffeomorphic to a 3-manifold W^3 which is obtained as follows: First attach finite handles of index 1 on a pairwise disjoint union $\cup \partial D_t^3$ of finite 3-disks $D_t^3 (t=1, \cdots, l)$, that is, let

$$h_i^1|\{\pm 1\}\times D^2 : \{\pm 1\}\times D^2 \longrightarrow \cup \partial D_t^3$$

be embeddings which have pairwise disjoint images on $\cup\partial D_t^3$, then attach all the handles of index 1 on $\cup\partial D_t^3$ by using the attaching maps

$$h_i^1|\{\pm 1\}\times D^2 : \{\pm 1\}\times D^2 \longrightarrow \cup \partial D_t^3$$

smoothly which is denoted by

$$N^3=(\bigcup_{t=1}^{l} D_t^3)\cup(\bigcup_{i=1}^{m} h_i^1).$$

And then by attaching finite handles of index 2 on ∂N^3, that is, let

$$h_i^2|\partial D^2\times D^1 : \partial D^2\times D^1 \longrightarrow \partial N^3$$

be embeddings which have pairwise disjoint images on ∂N^3, then attach all the handles of index 2 on ∂N^3 by using the attaching maps

$$h_i^2|\partial D^2\times D^1 : \partial D^2\times D^1 \longrightarrow \partial N^3$$

smoothly which is denoted by

$$N^3\cup(\bigcup_{j=1}^{n} h_j^2)=(\bigcup_{t=1}^{l} D_t^3)\cup(\bigcup_{i=1}^{m} h_i^1)\cup(\bigcup_{j=1}^{n} h_j^2).$$

And at last the 3-manifold W^3 which is obtained by attaching finite handles of index 3 on $\partial(N^3\cup(\cup h_j^2))$ smoothly:

$$W^3=(\bigcup_{t=1}^{l} D_t^3)\cup(\bigcup_{i=1}^{m} h_i^1)\cup(\bigcup_{j=1}^{n} h_j^2)\cup(\bigcup_{k=1}^{p} h_k^3).$$

Thus if V^3 is diffeomorphic to W^3 which is constructed by attaching handles succesively in order of small indexes of handles, it is said that V^3 has a normalized handle-decomposition.

Theorem 3.8 [6]. *Let W^3 be any connected, compact 3-manifold. Then there is a normalized handle-decomposition.*

Theorem 3.9. *Let M^3 be a simply-connected, 3-manifold with $\partial M^3=S^2$. Suppose that for any finite set of knots $K_i(i=1, \cdots, n)$ in M^3 such that $\partial K_i\subset\partial M^3$ there is a collar neighborhood V^3 of ∂M^3 in M^3 with $K_i\subset V^3(i=1, \cdots, n)$. Then M^3 is diffeomorphic to a 3-disk D^3.*

Proof. Now let $N^3=D^3\cup M^3$ and N^3 has the following handle-decomposition:

$$N^3=D^3\cup M^3=D^3\cup(\bigcup_{i=1}^{n} h_i^1)\cup(\bigcup_{i=1}^{n} h_i^2)\cup D^3.$$

And then it may be assumed that

$$M^3=D^3\cup(\bigcup_{i=1}^{n} h_i^1)\cup(\bigcup_{i=1}^{n} h_i^2)$$

where each embedding

$$h_i^1 : [-1, 1]\times D^2 \longrightarrow N^3 \quad (i=1, \cdots, n)$$

is a handle of index 1, and each embedding

$$h_i^2 : D^2\times[-1, 1] \longrightarrow N^3 \quad (i=1, \cdots, n)$$

is a handle of index 2.

Now let

$$K_i=h_i^2(\{0\}\times[-1, 1]) \quad (i=1, \cdots, n).$$

Then by our assumption there is a collar neighborhood V^3 of ∂M^3 such that

$$K_i\subset V^3 \quad (i=1, \cdots, n).$$

And $W^3=M^3-V^3$ is a simply-connected 3-manifold whose boundary is a 2-sphere.

Now let

$$H^3 = M^3 - \bigcup_{i=1}^{n} \mathrm{Int}\, N(K_i)$$

where $\{N(K_i)\ ;\ i=1, 2, \cdots, n\}$ consists of pairwise disjoint closed tubular neighborhoods of K_i in M^3. Then since H^3 is a handlebody of genus n, therefore H^3 is embeddable into R^3 :

$$W^3 \subset H^3 \subset R^3.$$

Therefore it follows by using the 3-dimensional Schönflies Theorem that W^3 is a 3-disk in R^3. Hence $M^3 = V^3 \cup W^3$ is a 3-disk. This completes the proof of Theorem 3.9.

Note that R. H. Bing proved the following Theorem [1] :

Theorem. *Let M^3 be a closed 3-manifold. Suppose that for any knot K in M^3 there is a 3-disk which contains K, then M^3 is homeomorphic to a 3-sphere.*

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