The estimation of volumes of handlebodies by inequalities in a simply-connected and closed three manifold

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1. Introduction

In this note I shall discuss with the category of smoothness for manifolds and mappings and let M^3 be a simply-connected and closed 3-manifold. Let $f: M^3 \longrightarrow R^7$ be an embedding and the Riemannian metric ρ of M^3 is the induced one from the standard inner product in R^7 through f. The length and the area of a curve L and a disk D^2 in M^3 are denoted with $\rho(L)$ and $s(D^2)$ respectively. And set $S= \inf\{s(D^2)\ ;\ D^2$ runs over all the meriodional 2-disks of $H^3\}$, $C= \inf\{\rho(L)\ ;\ L$ runs over all the longitudes of $H^3\}$. And let $v(H^3)$ be a volume of H^3. I shall estimate volumes of handlebodies by inequalities in the manifold M^3, that is, the following main Theorem 18 shall be proved in this note:

Theorem 18. *Let H^3 be a handlebody of genus 1 in the manifold M^3. And let C be the infimum of the lengths of all the longitudes of H^3 and S the infimum of the areas of all the meridional 2-disks of H^3. Then $C \cdot S \leqq v(H^3)$.*

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2. Preliminaries.

In order to prove the main Theorem 18 I shall refer to [1].

Theorem 1. *Let* K *be a compact subset of a separable complete metric space* R. *If* $\{L_i \subset \mathrm{K}\ ;\ i=1, 2, \cdots\}$ *is parametrized by equicontinuous maps, then there is a converging subsequence in* $\{L_i \subset \mathrm{K}\ ;\ i=1, 2, \cdots\}$.

Let ρ be a Riemanian metric in M^3 and $f_n, f: \mathrm{I} \longrightarrow M^3$ continuous functions. And define $d(f_n, f) = \sup_{t \in I} \rho(f_n(t), f(t))$. Then in Theorem 1, f_n converges to f if

$$\lim_{n \to \infty} d(f_n, f) = 0.$$

Theorem 2. *If curves of finite lengths are bounded, then these curves are parametrized by equicontinuous maps.*

Theorem 3. *A sequence of curves $L_n (n = 1, 2, \cdots)$ converges to L, then the length of L is no less than the upper limit of these curves.*

Theorem 4. *If any two points in a compact set* K *are connected by a continuous map in* K, *then there is the curve having the minimal length among these curves.*

Theorem 5. *Let M^n be a compact, Hausdorff manifold of dimension n. Then there is an embedding of M^n into R^{2n+1}.*

Lemma 6 [3]. *Let M^m, N^n be compact manifolds and $f, g: M^m \longrightarrow N^n$ be smooth maps. Let d be a metric of N^n and N^n be embedded into R^k. Then there is a positive number $\varepsilon > 0$ such that f, g are homotopic if $d(f(p), g(p)) < \varepsilon$ for any $p \in M^m$.*

Theorem 7 [2]. *Let M^m, V^n be manifolds with $\partial V^n = \phi$ and $\delta: M^m \longrightarrow (0, \infty)$ be a positive valued continuous function. And suppose that there is a metric in V^n and that there is a continuous function*

$$f: M^m \longrightarrow V^n$$

and f is a 1:1-immersion on an open neighborhood of a closed subset N of M^m. If $n \geqq 2m+1$, then there is a 1:1-immersion

$$g: M^m \longrightarrow V^n$$

therefore if M^m is compact, then g is an embedding such that

(i) *g is a δ-approximation of f.*

(ii) *g is homotopic to f relative N.*

(iii) $f \mid N = g \mid N$.

Definition 8. Let $f: S^1 \times D^2 \longrightarrow M^3$ *be an embedding. Then $f(S^1 \times D^2) = H^3$ is called to be a handlebody of genus one. Let $g: D^2 \longrightarrow H^3$ be a proper embedding, then $g(D^2)$ is called a meridional 2-disk of H^3. Let $h_t: S^1 \longrightarrow M^3$ be an isotopy with $h_t = f \mid S^1 \times 0$ for all $t \leqq 0$ and $h_t(S^1) \subset H^3$ for all t. Then $h_1(S^1)$ is said a longitude of H^3.*

Now from Theorem5 there is an embedding $f: M^3 \longrightarrow R^7$. The Riemannian metric ρ of M^3 is the induced metric from the standard inner product in R^7 through f. Now from [4] we refer to the following Definitions and Theorems: Let K be a simplicial complex which is rectilinearly embedded in R^p. If σ is a closed simplex of K, then the star $\mathrm{St}(\sigma, K)$ of σ in K is the union of all the simplexes of K which contain σ. If x is any point in K, then define $\mathrm{St}(x, K)$ as $\mathrm{St}(\sigma, K)$ where σ is the simplex of K which contains x in its interior. Now let $f: K \longrightarrow M^n$ be a C^r-map, where M^n is any C^r-manifold of dimension n. That is, for each closed simplex σ of K, $f \mid \sigma$ has a C^r-extension in some neighborhood of σ in R^p. Then a picewise linear map

$$f_x : \mathrm{St}(x, K) \longrightarrow M^n_{fx}$$

is defined by $f_x(v) = (f \mid \sigma)_x(v)$ if $v \in \sigma \subset \mathrm{St}(x, K)$ and $x \in \sigma$ whee $(f \mid \sigma)_x$ is a differential of $f \mid \sigma$ at x and M^n_{fx} is the tangent space of M^n at $f(x)$. The map f is said *regular* if and only if f_x is a 1-1 into map for any $x \in K$ and f is said *nonsingular* if and only if it is a regular homeomorphism onto $f(K)$. A nonsingular homeomorphism of K onto M^n is said a *C^r-triangulation* of M^n.

Now the following Theorem is obtained from [3], [4] :

Theorem 9. *There is a C^r-triangulation for any C^r-manifold.*

Now from Theorem 9 above there is a C^r-triangulation for the 3-manifold M^3. Let (K, π, M^3) be a C^r-triangulation for M^3 where K is a rectilinear simplicial complex in R^7 in this case (see [4] for details). Then since $\pi : K \longrightarrow M^3$ is a homeomorphism of K onto M^3, there is an inverse homeomorphism $\pi^{-1} : M^3 \longrightarrow K \subset R^7$. Now let $g = j\pi^{-1} : M^3 \longrightarrow R^7$ where $j : K \longrightarrow R^7$ is an inclusion mapping. It is clear that $g \mid \mathrm{Int}\pi(\sigma^3)$ is a C^r-embedding for any 3-simplex σ^3 of K and g is not of class C^r at any point of the 2-skeleton of K for all $r \geqq 1$. Now by reforming g in any neighborhood of K_2 in M^3 where K_2 is the 2-skeleton of K as follows : Let σ^2 be any 2- simplex of K. For the brevity of notations we use the same notations σ as simplexes of M^3 in place of $\pi(\sigma)$. On considering σ^2 of any 2-simplex of M^3 let $N(\partial\sigma^2)$ be a suitable collar neighborhood in σ^2. Then reform g on any bicollar neighborhood $N(\sigma^2\text{-}N(\partial\sigma^2))$ of $\sigma^2\text{-}N(\partial\sigma^2)$ in M^3 by using the following C^r-maps : For the sake of brevity assume that $g(\sigma^2)$ is contained into the (x, y)-plane and then there are two 3-simplexes λ^3 and μ^3 such that $\lambda^3 \cap \mu^3 = \sigma^2$, that is, λ^3 and μ^3 have σ^2 as a common 2-face of λ^3 and μ^3. And assume that $g(\lambda^3)$ is contained into the (x, y, z)-subspace and $g(\mu^3)$ is contained into the (x, y, w)- subspace in R^7. Now let $s = h(t)$ be a smooth monotone increasing function such that

$$0 \leqq h(t) \leqq 1 \text{ for all } t \in R,\ h(t) = 1 \text{ for } t \geqq \varepsilon, \text{ and } h(t) = 0 \text{ for } t \leqq -\varepsilon/2$$

where ε is a positive small number. Then define $k(t) = h(-t)$ for $t \in R$. Now by using h and k for $g(p) = (x, y, z, w)$ for any point $p \in \lambda^3 \cup \mu^3$ define $g_1(p) = (x, y, k(z)z, h(w)w)$ where $g_1(\lambda^3 \cup \mu^3)$ is contained into the (x, y, z, w)-subspace of R^7. Then it is clear that on considering the rank of the differential g_{1*}, $g_1 \mid \mathrm{Int}(\lambda^3 \cup \mu^3)$ is a C^r-embedding into R^7. Now by doing the same procedure for all the 2-simplexes of M^3 for the map $g_1 : M^3 \longrightarrow R^7$ we may assume that the restricted map of g_1 on $M^3\text{-Int}N(\pi K_1)$ is a C^r-embedding into R^7 where K_1 is the 1- skeleton of K and $N(\pi K_1)$ is a suitable closed neighborhood of πK_1 in M^3, therefore we may assume that $N(\pi K_1)$ is a handlebody of genus m and we use the same notation g_1 for the brevity of notations. Now by using Theorems 16, 17 for the handlebody of genus m in M^3 and the C^r-embedding

$$g_1 \mid \partial N(\pi K_1) : \partial N(\pi K_1) \longrightarrow R^7,$$

that is, and then on considering a suitable smooth tubular neighborhood for any 1-simplex in M^3 and by choosing suitable handles $H^3_i(i = 1, \cdots, n)$ of dimension 3 whose boundaries consist of all the longitudes of the handles, which are contained into $\partial N(\pi K_1)$, then we may assume that g_1 is of class C^r on the suitable tubular neighborhoods

$$H_i^3 (i = 1, \cdots, n)$$

of all the 1-simplexes by moving g_1 smoothly on all the handles $H_i^3 (i = 1, \cdots, n)$ which are suitable tubular neighborhoods of all the 1-simplexes, that is, such that

$$g_1 \mid H_i^3 : H_i^3 \longrightarrow II_i^3 \subset R^7$$

is a C^r-embedding for all i by choosing II_i^3 suitably with $g_1(\partial H_i^3) = \partial II_i^3$ for all i. Then after perfoming the same procedure for all the 1-simplexes of M^3 we may assume that

$$g_1 \mid ((M^3\text{-Int}N(\pi K_1)) \cup H_i^3) : (M^3\text{-Int}N(\pi K_1)) \cup H_i^3 \longrightarrow R^7$$

is a C^r-embedding, therefore there are 3-disks $D_i^3 (i = 1, \cdots, p)$ in M^3 whose centers consist of all the vertexes $\pi(K_0)$ of $\pi(K)$ such that $g_1 \mid M^3 - \cup \text{Int}D_i^3$ is a C^r-embedding where K_0 is the 0-skeleton of K. At last choosing suitable smooth 3-disks B_i^3 in R^7 for all i such that $\partial B_i^3 = g_1(\partial D_i^3)$ for $i = 1, \cdots, p$, then we may assume that g_1 is a C^r- embedding on all the 3-disks D_i^3 by moving g_1 smoothly on a suitable tubular neighborhood of any 0-simplex, that is, such that $g_1 \mid D_i^3 : D_i^3 \longrightarrow B_i^3$ is a C^r- embedding for all i.

Now replace g_1 with f throughout this note. Then from the discussions above it is clear that $f: M^3 \longrightarrow R^7$ is a C^r-embedding and for each σ^3, $f(\sigma^3\text{-}N(\partial\sigma^3))$ is contained into some three dimensional subspace of R^7.

Theorem 10. *Let M^n be a manifold with $\partial M^n \neq \phi$. Then there is a collar neighbor hood of ∂M^n.*

Proof. Refering to [2], let $\pi: TM^n \longrightarrow M^n$ be a tangent bundle. Let V be an open neighborhood of $\{(p, v) ; p \in \text{Int}M^3$, or $p \in \partial M^3$ and v is an inward vector$\}$ in TM^3. Then define an exponential map $exp: V \longrightarrow M^3$ as $exp(p, v) = \gamma_p(1)$ for any $(p, v) \in V$ where $\gamma_p(t)$ is a geodesic with $\gamma_p(0) = p$ and $\dfrac{d\gamma_p(0)}{dt} = v$.

Now there is a unique vector $X(p) \in TM^3$ for any point $p \in \partial M^3$ such that the following conditions (1), (2), (3) are satisfied:

(1) $X(p)$ is an inward vector at a point p in M^3.

(2) $\mid X(p) \mid = 1$.

(3) $X(p)$ is perpendicular to any vector of $T_p(\partial M^3)$.

Set $W = \{(p, s) ; p \in \partial M^3.\ 0 \leqq s < \infty,\ sX(p) \in V\}$. Then W is an open subset of $\partial M^3 \times [0, \infty]$. Define a map

$$k_1 : W \longrightarrow M^3$$

by $k_1(p. s) = exp(sX(p))$. Then k_1 is smooth, $k_1(p, 0) = p$, $(k_1)_*$ is a linear isomorphism and $\dfrac{\partial k_1(p, 0)}{\partial s} = p$. Therefore by the Inverse Function Theorem there is an open neighborhood $V^1(p)$ at $(p, 0)$ in W such that

$$k_1 \mid V^1(p) : V^1(p) \longrightarrow k_1(V^1(p))$$

is a diffeomorphism. And there is a smooth map $\varepsilon : \partial M^3 \longrightarrow (0, \infty)$ such that k_1 $\{(p, s) ; p \in \partial M^3.\ 0 \leqq s \leqq \varepsilon(p)\}$ is a 1: 1-map. Now by using ε define a smooth map

$$k : \partial M^3 \times I \longrightarrow M^3$$

by $k(p, s) = k_1(p, s\varepsilon(p))$. Then k is a collar of ∂M^3.

Definition 11. *Let* M^n *be a manifold with nonempty boundary. If there is an ambient isotopy* $F: M^n \times R \longrightarrow M^n \times R$ *for collars* $g, g' : \partial M^n \times I \longrightarrow M^n$ *such that* $F(p, 0) = (p, 0)$ *for any* $p \in \partial M^3$, *and by setting* $F(p, t) = (f_t(p), t)$, g, g' *satisfy that* $g' = f_1 g$, *then* g, g' *are said equivalent.*

Theorem 12. *If* ∂M^n *is compact, any collar of* ∂M^n *is equivalent.*

Definition 13. *Let* W^q, M^n *be manifolds and* $g_t : W^q \longrightarrow M^n$ *be an isotopy with* g_t *embeddings for* $t = 0, 1$. *Suppose that there is an ambient isotopy* $F: M^n \times R \longrightarrow M^n \times R$ *such that by setting* $F(p, t) = (f_t(p), t)$, f_0 *is an identity,* $f_t(g_0(p)) = g_t(p)$ *for all* $p \in W^q$, $t \in R$. *Then* F *is said an ambient isotopy enclosing* g_t.

Theorem 14 [2] (The Isotopy Extesion Theorem). *Let* W^q, M^n *be manifolds and* $G: W^q \times R \longrightarrow M^n \times R$ *be an isotopy beween embeddings* g_i *for* $i = 0, 1$. *If supp* G *is compact, then there is an ambient isotopy* $F: M^n \times R \longrightarrow M^n \times R$ *which encloses* G *and moreover supp*F *is compact.*

Remark. To see the existence of an ambient isotopy

$$F: M^n \times R \longrightarrow M^n \times R$$

that encloses G, note that

$$G(W^q \times R)$$

is a submanifold of $M^n \times R$, and construct a vector field

$$X: M^n \times R \longrightarrow T(M^n \times R) = TM^n + TR$$

as follows: by setting $G(p, t) = (g_t(p), t)$, define a vector field

$$Y: G(W^q \times R) \longrightarrow TG(W^q \times R)$$

by $Y(G(p. t)) = (\frac{\partial g_t(p)}{\partial t}, 1)$. And also define a vector field

$$Z: M^n \times R \longrightarrow T(M^n \times R)$$

by $Z(x. t) = (0_x, 1)$.

Now let $\{\varphi_Y, \varphi_Z\}$ be a partition of unity for an open covering $\{U_Y, U_Z\}$ of $M^n \times R$ with $G((\mathrm{supp}G) \times I) \subset U_Y$, $M^n \times R - G((\mathrm{supp}G) \times I) \subset U_Z$. Now define a vector field

$$X: M^n \times R \longrightarrow T(M^n \times R) = TM^n + TR$$

by setting $X(x, t) = \varphi_Y(x, t)(\frac{\partial g_t(p)}{\partial t}, 1) + \varphi_Z(x, t)(0_x, 1)$. *Note that*

$$g_t(p)\ (0 \leqq t \leqq 1)$$

is an integral curve of the vector field Y.

Definition 15. Let W^q *be a submanifold of an* n*-manifold* M^n *with* $\angle W^q = \phi$ *and* $\nu(M^n/W^q) = \{E(\nu), W^q, \pi(\nu)\}$ *be a normal bundle of* W^q. *By introducing a Riemannian metric on each fiber of* $\nu(M^n/W^q)$, *set* $NE(\nu) = \{v \in E(\nu) : |v| \leqq 1\}$. *Now let* $g, g' : NE(\nu) \longrightarrow M^n$ *be tubular neighborhoods of* W^q. *Then* g, g' *are said* *equivalent if there is an ambient isotopy* $F: M^n \times R \longrightarrow M^n \times R$ *such that the following conditions are satisfied* :

(i) F *fixes each point of* W^q.

(ii) *Setting* $F(p, t) = (f_t(p), t)$, *then we have* $g'(NE(\nu)) = f_1 g(NE(\nu))$.

(iii) *There is a bundle mapping* $\chi: E(\nu) \longrightarrow E(\nu)$ *satisfying the following conditions* :

(a) *The mapping between the base space of* $E(\nu)$, *which is determined by* χ, *is an identity.*

(b) χ *maps each fiber itself and preserves the Riemannian metric, that is,*

$$(\chi(v), \chi(v'))_b = (v, v')_b \text{ for all } v, v' \in \pi^{-1}(b).$$

Then we see that $\chi \mid NE(\nu) : NE(\nu) \longrightarrow NE(\nu)$ *is smooth.*

(c) $g'\chi \mid NE(\nu) = f_1 g$.

Theorem 16 [2] (The Uniqueness Theorem of Tubular Neighborhoods). *Let* W^q *be a submanifold of* M^n *with* $\angle W^q = \phi$. *If* W^q *is compact, then any tubular neighborhood of* W^q *is equivalent. And then by Theorem 14 the enclosing ambient isotopy has a compact support.*

Now we shall give a proof of Theorem 16 in case of a 1-closed submanifold in M^3 as follows:

Theorem 17. *Let* K *be a closed 1-submanifold of* M^3. *Suppose that* $g, g' : R^2 \times S^1 \longrightarrow M^3$ *are tubular neighborhoods of* K *in* M^3. *Then* g *and* g' *are equivalent.*

Proof. Note that $g(0, z) = g'(0, z) \in K$ for all $z \in S^1$ and all the surfaces

$$\{g^{-1}g'(R^2, z) \;;\; \forall z \in S^1,\ g^{-1}g' : R^2 \times S^1 \longrightarrow R^2 \times S^1\}$$

consist of pairwise disjoint ones F_z in $R^2 \times S^1$.

The normal vector at $(0, 0, z)$ of the surface F_z is given by

$$n_z = \left(\frac{\partial y'}{\partial x}\frac{\partial z'}{\partial y} - \frac{\partial y'}{\partial y}\frac{\partial z'}{\partial x},\ \frac{\partial z'}{\partial x}\frac{\partial x'}{\partial y} - \frac{\partial z'}{\partial y}\frac{\partial x'}{\partial x},\ \frac{\partial x'}{\partial x}\frac{\partial y'}{\partial y} - \frac{\partial x'}{\partial y}\frac{\partial y'}{\partial x}\right).$$

Since each surface $F_z (\forall z \in S^1)$ transverses at $(0, 0, z)$ for S^1, the third component of n_z is nonzero, therefore we may assume that the third component is positive and moreover $|n_z| = 1$ for a time. Then the family of all the surfaces $G_z(t)$ $(\forall z \in S^1)$ which have the system of the vectors $(1-t)n_z + te_z (0 \leqq t \leqq 1)$ as the normal vectors defines an isotopy on a slenderly tubuler neighborhood of $0 \times S^1$. Because in order to construct our desired isotopy between $F_z (\forall z \in S^1)$ and $e_z (\forall z \in S^1)$ set (x', y', z'') that is determined by the surface $G_z(t)$, that is, perpendicular to $(1-t)n_z + te_z (0 \leqq t \leqq 1)$, for each $z \in S^1$ where $z'' = ax' + by' + c$ for some real numbers a, b, c.

Now two maps (x', y', z') and (x', y', z'') have the same normal vector at

$(0, 0, z)$. And we have that $\frac{\partial z'}{\partial z}\Big|_{x=y=0} = \frac{\partial z''}{\partial z}\Big|_{x=y=0} = 1$. Therefore an isotopy between these maps is given by the map $(x', y', (1-t)z' + tz'')$. Now by using the isotopy (*) suitably and the Isotopy Extension Theorem 14 we may assume that $F_z = G_z(0)$.

Now by using all the surfaces $G_z(t)$ $(\forall z \in S^1)$ and suitable tubular neighborhoods we shall construct an isotopy between $g^{-1}g'$ and $G_z(1)$ where $G_z(1)$ is the $(x. y)$-plane having a height z in $R^2 \times S^1$. Therefore there is a map $k : N(S^1) \longrightarrow R^2 \times S^1$ by defining $k(x, y, z) = (x', y', z)$ where $N(S^1)$ is a suitable tubular neighborhood of $0 \times S^1$ as the result that has used the isotopy above. That is to say, since $g^{-1}g'(x, y, z) = (x', y', z')$ is a diffeomorphism and $\frac{\partial x'}{\partial x}\frac{\partial y'}{\partial y} - \frac{\partial x'}{\partial y}\frac{\partial y'}{\partial x} > 0$ by our assumption, therefore define a smooth map

$$k : N(S^1) \longrightarrow R^2 \times S^1$$

by $k(x, y, z) = (x', y', z)$ easily. Moreover by using a suitable isotopy (*) of $N(S^1)$ in $R^2 \times S^1$ we may assume that the domain of k is $R^2 \times S^1$ and similarily by using (*)

$$k(R^2 \times S^1) = R^2 \times S^1.$$

By using the polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, an isotopy that maps the unit open disk Int D^2 onto R^2 is given as follows:

$$(*) \qquad f_t(r, \theta) = ((1-t)x + \frac{tx}{1-r}, (1-t)y + \frac{ty}{1-r})$$

where the Jacobian is $J(f_t) = r(1-t + \frac{t}{1-r})(1-t + \frac{t}{1-r} + \frac{tr}{(1-r)^2}) > 0$.

And the map $(x', y', tz + (1-t)z')$ is an isotopy on some tubular neighborhood of $0 \times S^1$ between $g^{-1}g'$ and k. Now since $k(0, 0, z) = (0, 0, z)$, we have that

$$\frac{\partial x'}{\partial z}\Big|_{x=y=0} = \frac{\partial y'}{\partial z}\Big|_{x=y=0} = 0.$$

Therefore we can define a bundle map $\chi_1' : R^2 \times S^1 \longrightarrow R^2 \times S^1$ between the bundles $R^2 \times S^1 \longrightarrow S^1$ by $\chi_1'(x, y, z) = k(x, y, z)$. Then an isotopy of $\chi_1'F : R^2 \times S^1 \times [0. 1] \longrightarrow R^2 \times S^1 \times [0. 1]$ is given as follows:

$$F(x, y, z, t) = (\frac{x'(tx, ty, z)}{t}, \frac{y'(tx, ty, z)}{t}, z) \qquad (0<t<1),$$

$$F(x, y, z, 0) = \lim_{t \to 0}(\frac{x'(tx, ty, z)}{t}, \frac{y'(tx, ty, z)}{t}, z) \qquad (t = 0).$$

Now from the formuae above we have that $F(0, 0, z, t) = (0, 0, z)$ for all t. Therefore $F(x, y, z, 0)$ is given as follows:

$$F(x, y, z, 0) = (\begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, z).$$

Now since $a(z)d(z) - b(z)c(z) > 0$ for all $z \in S^1$, hence by the Schmidt's

Orthogonalization Theorem for $(a(z), b(z)), (c(z), d(z))$, let

$$\{(\alpha(z), \beta(z)), (\gamma(z), \delta(z))\}$$

be the orthogonalized vectors. Then we have the following:

$$\begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix} = \begin{pmatrix} \alpha(z) & \beta(z) \\ \gamma(z) & \delta(z) \end{pmatrix} \begin{pmatrix} \lambda(z) & \mu(z) \\ 0 & \nu(z) \end{pmatrix}$$

where $\lambda(z)$, $\mu(z)$ and $\nu(z)$ are smooth.

Now set

$$\chi = \begin{pmatrix} \alpha(z) & \beta(z) \\ \gamma(z) & \delta(z) \end{pmatrix}$$

And moreover define as follows:

$$h_t = \chi \begin{pmatrix} (1-t)\lambda(z) + t & (1-t)\mu(z) \\ 0 & (1-t)\nu(z) + t \end{pmatrix}$$

where $h_t (0 \leqq t \leqq 1)$ induces a bundle map with $h_0 = \chi_0$, and $h_1 = \chi$ preserves the Riemannian metric on each fiber. From the discussions above $g^{-1}g'$ and χ are isotopic. Hence by using the Isotopy Extension Theorem 14 we can see that g' and $g \circ \chi$ are equivalent.

3. Proofs of the main Theorems

Theorem 18. *Let H^3 be a handlebody of genus 1 which is embedded into the closed manifold M^3. And let C be the infimum of the lengths of all the longitudes of H^3 and S the infimum of the areas of all the meridional 2-disks of H^3. Then $C \cdot S \leqq v(H^3)$.*

Proof. Note that $C > 0$ because there is a C^r-triangulation for the three dimensional manifold M^3 from the preceding discussions.

Set $D_0^2 = \{(x, y, 0) \in R^3 ; x^2 + (y-b)^2 \leqq a^2, 0 < a < b\}$. Then rotating the 2-disk D_0^2 around x-axis, a handlebody II^3 of genus 1 is obtained in R^3. Let $g : II^3 \longrightarrow H^3$ be a diffeomorphism. Now let L be a longitude of H^3 with $L \subset \mathrm{Int}H^3$. And then we construct a tubular neighborhood $N(L) \subset \mathrm{Int}H^3$ of L as follows: Let $\mathbf{N}(L) \longrightarrow L$ be a normal vector bundle. By using the exponential map $exp : \mathbf{N}(L) \longrightarrow M^3$, set

$$N(L) = \{exp(p, v) : |v| < \varepsilon, p \in L\}$$

for a sufficiently small positive number ε. Now

$$N(L) \text{ and } H^3$$

are both tubular neighborhoods of L, therefore from Theorems 16, 17

$$N(L) \text{ and } H^3$$

are equivalent.

Now by using the Isotopy Extension Theorem 14 for $N(L)$ and H^3, there is an ambient isotopy $g_t : M^3 \longrightarrow M^3$ such that g_0 is an identity and

$$g_1(N(L)) = H^3.$$

Now setting $II^3 = N(L)$ we may use the diffeomorphism $g = g_1$ henceforth.

Now in order to divide H^3 first divide II^3 as follows: Divide the disk D_0^2 by all the straight lines $y = x \tan \frac{m\pi}{2^n} + b$, $x = 0$ and all the circles

$$x^2 + (y-b)^2 = (\frac{ma}{2^n})^2 \ (n = 0, 1, \cdots, m = 0, 1, \cdots, 2^n).$$

And moreover by rotating the divided disk around x-axis and by dividing the semi-divided handlebody II^3 with all the planes $z = y \tan \frac{m\pi}{2^n}$, $y = 0$ for $n = 0, 1, \cdots, m = 0, 1, \cdots, 2^n$ we have the divided handlebody II^3 and by using g we have at last the divided handlebody H^3. Let $\omega^{(n)}{}_{ijm}$ be the small regions that are obtained by dividing H^3 thus for each n. Then we have:

$$\lim_{n\to\infty} \sum_{i,j,m} v\ (\omega^{(n)}{}_{ijm}) = v(H^3).$$

Now let $D^2{}_{nm}$ be the sliced disk of II^3 by $z = y \tan \frac{m\pi}{2^n}$, $y = 0$ for $n = 0, 1, \cdots, m = 0, 1, \cdots, 2^n$. By seeing the proof of Theorem 10,take a suitable collar neighborhood $N(\partial H^3)$ of ∂H^3 in H^3. By using the Isotopy Extension Theorem 14 for the two collar neighborhoods $N(\partial H^3)$ and $g \mid S^1 \times \partial D_0^2 \times [0, \varepsilon)$, we may assume that each meridional 2-disk $g(t \times D^2)$ is perpendicular to ∂H^3 along $g(t \times \partial D^2)$ for all t where we identify II^3 with $S^1 \times D^2$ and set $S^1 = I/\ \{0, 1\}$.

Now divide $\partial D_0^2 = \{(x, y) \in R^2;\ x^2 + (y-b)^2 = a^2, 0 < a < b\}$ with all the straight lines $y = x \tan \frac{m\pi}{2^n} + b$, $x = 0$ for $n = 0, 1, \cdots, m = 0, 1, \cdots, 2^n$ and let $\{T_{nm} \subset \partial D^2{}_0;\ m = 0, 1, \cdots, 2^n\}$ be the divided arcs. By using Theorems 16, 17 for each arc $g(t \times (T_{nm} \cup T_{nm+1})) \subset \partial H^3$ and each embedding $\sigma_m : S^1 \times (N(\partial T_{nm} \cap \partial T_{nm+1})) \longrightarrow g(S^1 \times (T_{nm} \cup T_{nm+1})$ such that $\sigma_m(t, q) = g(t, q)$ for all the points $(t, q) \in S^1 \times (N\partial(T_{nm} \cup T_{nm+1}))$ and $g(S^1 \times (\partial T_{nm} \cup \partial T_{nm+1}))$ is perpendicular to $\sigma_m(t \times N(\partial T_{nm} \cap \partial T_{nm+1}))$ for $m = 0, 1, \cdots, 2^n$ and for all $t \in S^1$ where $N(\partial T_{nm} \cap \partial T_{nm+1})$ is a tubular neighborhood of $\partial T_{nm} \cap \partial T_{nm+1}$ in $T_{nm} \cup T_{nm+1}$ and $N\partial(T_{nm} \cup T_{nm+1})$ is a collar of $\partial(T_{nm} \cup T_{nm+1})$ in $T_{nm} \cup T_{nm+1}$, as a result $g \mid S^1 \times (T_{nm} \cup T_{nm+1})$ and σ_m are isotopic, that is, equivalent in $T_{nm} \cup T_{nm+1}$, then as $\chi = 1$ in Definition 15 and then from Theorem 14 there is a suitable enclosing ambient isotopy

$$\tilde{G} : g(S^1 \times (T_{nm} \cup T_{nm+1})) \times R \longrightarrow g(S^1 \times (T_{nm} \cup T_{nm+1})) \times R$$

between $g \mid S^1 \times (T_{nm} \cup T_{nm+1})$ and σ_m such that $\tilde{G}$ fixes all the points of $g(S^1 \times \partial(T_{nm} \cup T_{nm+1}))$ and setting $\tilde{G}(p, s) = (\tilde{g}_s(p), s)$ as $\chi = 1$, $\tilde{g}_1 \circ g \mid S^1 \times (N(\partial T_{nm} \cup \partial T_{nm+1})) = \sigma_m$. And perform the same method above for $m = 0, 1, \cdots, 2^n$.Then as a result we may assume that the closed curves $\tilde{g}_1 \circ g(S^1 \times \partial T_{nm}) = g(S^1 \times \partial T_{nm})$ are perpendicular to

$$\sigma_m(t \times \partial D_0^2) = \tilde{g}_1 \circ g(t \times \partial D_0^2) \text{ for any } t \in S^1 \text{ and for } m = 0, 1, \cdots, 2^n.$$

Moreover by using the same notations in the proof of Theorem10 and the isotopy

$$\tilde{g}_s \circ g \mid \partial II^3 : \partial II^3 \longrightarrow \partial H^3$$

we define a new isotopy

$$f_s'\circ(g \mid \partial D_0^2 \times i) : S^1\times\partial D_0^2 \times [0, \varepsilon] \longrightarrow H^3$$

by $f_s'(g(t, p), \tau) = k(\tilde{g}_s\circ g(t, p), \tau)$ for all $t\in S^1$, $p\in\partial D_0^2$ and $\tau\in [0, \varepsilon]$ where $\partial H^3 = S^1\times \partial D_0^2$, and ε is a small positive real number and then we assume that

$$k(\partial H^3\times [0,\varepsilon]) = g(S^1\times \partial D_0^2 \times [0, \varepsilon]) \text{ and } \partial D_0^2\times [0, \varepsilon] \subset D_0^2$$

from the preceding arguments. Then by using the Isotopy Extension Theorem 14 for the isotopy $f_s'\circ(g \mid \partial H^3\times i)$ there is a suitable enclosing ambient isotopy

$$F: H^3\times R \longrightarrow H^3\times R$$

such that setting $F(q, s) = (f_s(q), s)$, $f_s\circ f_0'(g(t, p), \tau) = f_s'(g(t, p),\tau)$ for any $s\in[0, 1]$ and $\tau\in[0, \varepsilon]$, in particular we have that $f_1\circ f_0'(g(t, p), \tau) = f_1'(g(t, p),\tau)$. Now for the sake of simplicity we replace $f_1\circ g$ with g_n. And let $h_n: \partial H^3 \longrightarrow I$ be the Morse function obtained from g_n by $h_n(p) = t$ where $g_n(t, q) = p$ for $t\in I$ and $q\in\partial D_0^2$. Since h_n has no critical point, we may use $\xi_n = \dfrac{\mathrm{grad}h_n}{\| \mathrm{grad}h_n \|^2}$ in place of $\mathrm{grad}h_n$, and also we may assume that $\mathrm{grad}h_n$ is equal to ξ_n henceforth. And let $\varphi_p^{(n)}(t)$ be the integral curve of the gradient vector field $\mathrm{grad}h_n = \xi_n$.

Now by using the integral curves $\varphi_p^{(n)}(t)$, for example, for any $t\in[0, 1-\frac{1}{2^n}]$ define a tubular neighborhood of $L_i (i = 1, 2)$ as $\{\varphi_p^{(n)}(t) ; p\in g_n(S_i), t\in[0, 1-\frac{1}{2^n}]\}$ where each $L_i = \varphi_{p_i}^{(n)}(I)$ is a longitude of ∂H^3 with

$$p_1 = g_n(0, a+b),\ p_2 = g_n(0, b-a)$$

and set $S_1 = \{(x, y)\in\partial D_0^2;\ y \geqq b\}$, $S_2 = \{(x, y)\in\partial D_0^2;\ y \leqq b\}$.

Now $g_n|([0, 1-\frac{1}{2^n}]\times S_i)$ and $\{\varphi_p^{(n)}(t) ; p\in g_n(S_i),\ t\in[0, 1-\frac{1}{2^n}]\}$ are isotopic for $i = 1, 2$ in ∂H^3 with fixing all the boundary points of these tubular neighborhoods, because these tubular neighborhoods of each L_i are equivalent in ∂H^3 with fixing all the boundary points of these tubular neighborhoods from Theorems 16, 17. And from Theorem14 and as $\chi = 1$ in Definition 15 there is an enclosing ambient isotopy

$$f_s: \partial H^3 \longrightarrow \partial H^3$$

for these pairs such that $f_1\circ g_n(t, q) = \varphi_p^{(n)}(t)$ for any $t\in[0, 1-\frac{1}{2^n}]$, $q\in\partial D_0^2$, and $g_n(0, q) = p$ with f_0 an identity. Then from Remark under Theorem 14 we may assume that $f_s\circ g_n(t, q) = g_n(t, q)$ for all the points $q\in\partial D_0^2$ and all $t\in[1-\frac{1}{2^n}+\varepsilon_n, 1]$ for $0<\varepsilon_n<\frac{1}{3\cdot 2^n}$. Moreover by using the same notations in the proof of Theorem 10 and the isotopy $f_s\circ g_n \mid \partial H^3: \partial H^3 \longrightarrow \partial H^3$ as before define a new isotopy

$$f_s'\circ(g_n \mid \partial H^3\times i) : \partial H^3\times [0, \varepsilon] \longrightarrow H^3$$

by $f_s'(g_n(t, q), \tau) = k(f_s\circ g_n(t, q), \tau)$ for all $t\in S^1$, $q\in\partial D_0^2$ and $\tau\in [0, \varepsilon]$ where we assume that $k(\partial H^3\times [0, \varepsilon]) = g_n(S^1\times \partial D_0^2\times [0, \varepsilon])$ and $\partial D_0^2\times [0, \varepsilon]\subset D_0^2$ as before.

Then by using the Isotopy Extension Theorem 14 for the isotopy $f_s' \circ (g_n \mid \partial H^3 \times i)$ there is a suitable enclosing ambient isotopy

$$\tilde{G}: H^3 \times R \longrightarrow H^3 \times R$$

such that setting $\tilde{G}(p, s) = (\tilde{g}_s(p), s)$, $\tilde{g}_s \circ f_0'(g_n(t, q), \tau) = f_s'(g_n(t, q), \tau)$ for any $s \in [0, 1]$ and $\tau \in [0, \varepsilon]$, in particular we have that $\tilde{g}_1 \circ f_0'(g_n(t, q), \tau) = f_1'(g_n(t, q), \tau)$.

Now for the sake of simplicity we replace $\tilde{g}_1 \circ g_n$ with g_n. Therefore from the arguments above we see that $g_n^{-1}(\varphi_p^{(n)}(t))$ is contained into $S_1 \times q$ where

$$g_n(0, q) = p \text{ and } 0 \leqq t \leqq 1 - \frac{1}{2^n}.$$

Now for $n+1$ the divided arcs $\{T_{n+1m}; m = 0, 1, \cdots, 2^{n+1}\}$ is a refinement of $\{T_{nm}; m = 0, 1, \cdots, 2^n\}$ in ∂D_0^2. Then perform the same method above for the refinement $\{T_{n+1m}; m = 0, 1, \cdots, 2^{n+1}\}$. Each arc T_{nm} is refined into two arcs in $\{T_{n+1m}; m = 0, 1, \cdots, 2^{n+1}\}$, for example, T_{nm} is refined into T_1 and T_2: $T_{nm} = T_1 \cup T_2$. Now the integral curves ∂T_{nm} are closed among all the integral curves of gradh_n in T_{nm}.

Now let $\sigma: S^1 \times (\partial T_1 \cap \partial T_2) \times I \longrightarrow g_n(S^1 \times (T_1 \cup T_2))$ be a tubular neighborhood such that $\sigma(S^1 \times (\partial T_1 \cap \partial T_2) \times I) \subset \mathrm{Int}\, g_n(S^1 \times (T_1 \cup T_2))$ and the curve $\sigma(t \times q \times I)$ is perpendicular to the closed curve

$$g_n(S^1 \times (\partial T_1 \cap \partial T_2)) \text{ and } \sigma \mid S^1 \times N(\partial T_{nm}) = g_n \mid S^1 \times N(\partial T_{nm})$$

where $N(\partial T_{nm})$ is a collar of ∂T_{nm} in T_{nm}.

Now σ and $g_n \mid S^1 \times (T_1 \cup T_2)$ are equivalent in $S^1 \times (T_1 \cup T_2)$ from Theorems 16, 17. Therefore from Theorem 14 and as $\chi = 1$ in Definition 15 there is an encosing ambient isotopy $f_s: S^1 \times T_{nm} \longrightarrow S^1 \times T_{nm}$ for the equivalent maps σ and $g_n \mid S^1 \times T_{nm}$ such that $f_1 \circ g_n \mid S^1 \times T_{nm} = \sigma$, the map $f_1 \circ g_n$ fixes all the points of $\partial T_1 \cup \partial T_2$ and $f_1 \circ g_n(t \times T_{nm})$ is perpendicular to all the closed curves $f_1 \circ g_n(S^1 \times (\partial T_1 \cup \partial T_2))$ in ∂H^3. Now perform the same method above for all the divided arcs $\{T_{n+1m}; m = 0, 1, \cdots, 2^{n+1}\}$. And we use g_n in place of $f_1 \circ g_n$ and let h_n be the Morse function obtained from g_n.

Now as before by using the integral curves $\varphi_p^{(n+1)}(t)$, for example, for any $t \in [0, 1 - \frac{1}{2^{n+1}}]$ define tubular neighborhoods of L_i as

$$\{\varphi_p^{(n+1)}(t); p \in g_n(S_i), t \in [0, 1 - \tfrac{1}{2^{n+1}}]\}$$

where each L_i is a longitude of ∂H^3 as before. Note that $\varphi_p^{(n+1)}(t) = \varphi_p^{(n)}(t)$ for any $t \in [0, 1 - \frac{1}{2^n}]$ and for all $p \in g_n(\partial D_0^2)$.

Now $g_n \mid ([0, 1 - \frac{1}{2^{n+1}}] \times S_i$ and $\{\varphi_p^{(n+1)}(t); p \in g_n(S_i), t \in [0, 1 - \frac{1}{2^{n+1}}]\}$ are isotopic for $i = 1, 2$ in ∂H^3, therefore these tubular neighborhoods are equivalent from Theorems 10, 16, 17. And from Theorem 14 and as $\chi = 1$ in Definition 15 there is an enclosing ambient isotopy $f_s: \partial H^3 \longrightarrow \partial H^3$ for these isotopies such that

$$f_1 \circ g_n(t, q) = \varphi_p^{(n+1)}(t)$$

where $t \in [0, 1 - \frac{1}{2^{n+1}}]$, $q \in \partial D_0^2$, and $g_n(0, q) = p$.

Then from Remark under Theorem 14 we may assume that $f_s \circ g_n(t, q) = g_n(t, q)$ for all the points $q \in \partial D_0^2$ and all $t \in [1-\frac{1}{2^{n+1}} + \varepsilon_{n+1}, 1]$ for $0 < \varepsilon_{n+1} < \frac{1}{3 \cdot 2^{n+1}}$. Moreover by using the same notations in the proof of Theorem 10 and the isotopy

$$f_s \circ g_n \mid \partial II^3 : \partial II^3 \longrightarrow \partial H^3$$

as the preceding discussions define an isotopy

$$f_s' \circ (g_n \mid \partial II^3 \times i) : \partial II^3 \times [0, \varepsilon] \longrightarrow H^3$$

by $f_s'(g_n(t, q), \tau) = k(f_s \circ g_n(t, q), \tau)$ for all $t \in S^1$, $q \in \partial D_0^2$ and $\tau \in [0, \varepsilon]$ where as before we assume that $k(\partial H^3 \times [0, \varepsilon]) = g_n(S^1 \times \partial D_0^2 \times [0, \varepsilon])$ and $\partial D_0^2 \times [0, \varepsilon] \subset D_0^2$. Then by using the Isotopy Extension Theorem 14 for the isotopy $f_s' \circ g \mid \partial II^3$ there is a suitable enclosing ambient isotopy

$$\tilde{G} : H^3 \times R \longrightarrow H^3 \times R$$

such that by setting $\tilde{G}(p, s) = (\tilde{g}_s(p), s)$,

$$\tilde{g}_s \circ f_0'(g_n(t, q, \tau) = f_s'(g_n(t, q), \tau)$$

for any $s, \tau \in [0, \varepsilon]$ and $q \in \partial D_0^2$, in particular we have that

$$\tilde{g}_1 \circ f_0' \circ g_n(t, q) = f_1' \circ g_n(t, q).$$

Now for the sake of simplicity we replace $\tilde{g}_1 \circ g_n$ with g_{n+1}. Let $\varphi_p^{(n+1)}(t)$ be an integral curve of the Morse function h_{n+1} obtained from g_{n+1}. Therefore from the arguments above we see that $g_{n+1}^{-1}(\varphi_p^{(n+1)}(t))$ is contained into $S^1 \times q$ where

$$g_n(0. q) = p \text{ and } 0 \leqq t \leqq 1 - \frac{1}{2^{n+1}}.$$

Now by repeating the preceding arguments above there are sequences $\{g_n ; n = 1, 2, \cdots\}$, $\{h_n ; n = 1, 2, \cdots\}$ and a familiy of the integral curves:

$$\varphi_p^{(n)}([0, 1-\frac{1}{2^n}]) \subset \partial H^3 \text{ for } n = 1, 2, \cdots$$

Now as the preceding discussions set $\xi_n = \frac{\text{grad} h_n}{\| \text{grad} h_n \|^2}$. If we set $\sigma(t) = h_n(\varphi_p^{(n)}(t))$, then $\frac{d\sigma(t)}{dt} = 1$, therefore we have that

$$\sigma(t) = t \text{ as } \sigma(0) = h_n(\varphi_p(0)) = h_n(p) = 0.$$

Now we have that $\varphi_p^{(m)}(t)$ for $m < n$ and $0 \leqq t \leqq 1 - \frac{1}{2^m}$ from the construction of the Morse functions h_n. Now lot $l_n : g(D_0^2) \longrightarrow R$ define as

$$l_n(p) = \int_0^1 \| \dot{\varphi}_p^{(n)}(t) \| dt,$$

then g_n and l_n are both uniformly continuous.

Now we extend the preceding arguments into H^3. First divide the 2-disk

$$D_0^2 = \{(x. y) \in R^2 ; x^2 + (y-b)^2 \leqq a^2, 0 < a < b\}$$

by using the straight lines $y = x \tan \frac{m\pi}{2^n} + b$, $x = 0$ and the circles

$$x^2 + (y-b)^2 = (\frac{ma}{2^n})^2 \text{ for } n = 0, 1, \cdots, m = 0, 1, \cdots, 2^n.$$

Now rotate the divided 2-disk D_0^2 around x-axis. And moreover divide the semi-divided handlebody $II^3 = S^1 \times D^2$ by the planes :

$$z = y \tan \frac{m\pi}{2^n}, \ y = 0 \text{ for } n = 0, 1, \cdots\cdots, \ m = 0, 1, \cdots, 2^n.$$

Let $S^{(n)}{}_{ij0}$ be the small regions which are obtained by dividing D^2_0 as above. In general all the small regions which do not contain $(0, 0) \in S^1 \times 0$ have four corners in the small regions of D^2_0.

Now set $B^2{}_{nm} = \{(x. y);\ x^2 + (y-b)^2 \leqq (\frac{ma}{2^n})^2\}$ and set $U^3{}_{nm} = S^1 \times \{(x. y)\ ;\ (\frac{(m-1)a}{2^n})^2 \leqq x^2 + (y-b)^2 \leqq (\frac{(m+1)a}{2^n})^2\}$. Then we have that $S^1 \times \partial B^2{}_{nm} \subset U^3{}_{nm}$ for $m = 1, 2, \cdots, 2^n-1$.

Now perform the same procedure for $g_n(S^1 \times \partial B^2{}_{nm})$ as did before for ∂H^3. Therefore by using Theorems 10, 16, 17 and 14 in each $g_n(U^3{}_{nm})$ we may assume that each $g_n(t \times \partial B^2{}_{nm})$ is perpendicular to $g_n(S^1 \times \partial T^{(m)}{}_{nk})$ for all k along its intersection where $T^{(m)}{}_{nk}$ is the divided arcs of $\partial B^2{}_{nm}$ that are divided by the same procedure as did before for ∂H^3, that is, divide $\partial B^2{}_{nm}$ with the straight lines $y = x \tan \frac{m\pi}{2^n} + b,\ x = 0$ for $m = 0, 1, \cdots, 2^n$ and let $T^{(m)}{}_{nk}$ be the arcs that are divided by all the straight lines above. Now from the discussions above we can see that

$$g_n(t \times D^2) \text{ and } g_n(S^1 \times \bigcup_{m,\,k} \partial T^{(m)}{}_{nm})$$

are perpendicular, and moreover $g_n(t \times D^2)$ and $g_n(S^1 \times q)$ are perpendicular, in H^3 for any $t \in S^1$ and any $q \in D^2_0$.

Now note that all the surfaces $g_n(S^1 \times \{(x. y)\ ;\ y = x \tan\theta\ \pi + b\})$ for some θ and $g_n(S^1 \times \{(0, y)\ ;\ y \in R\})$ are perpendicular to all the tori $g_n(S^1 \times \bigcup_m \partial B^2{}_{nm})$ along

$$g_n(S^1 \times \bigcup_k \partial T^{(m)}{}_{nk}) \text{ for } m = 1, 2, \cdots, 2^n.$$

And we revise g_n as follows : that is, let $\Pi^2{}_{ijk}$ be 2- disks on D^2_0 such that the center of each $\Pi^2{}_{ijk}$ is one of the four corners of $S^{(n)}{}_{ij0}$ for $k = 1, 2, 3, 4$ in general and the diameters of the $\Pi^2{}_{ijk}$ are sufficiently small such that each $\Pi^2{}_{ijk}$ doesn't contain any other corners.

Now there are annuluses $A_i^{(n)}$ in $\mathrm{Int}H^3 - \bigcup_{i,\,j,\,k} \mathrm{Int}g_n(S^1 \times \Pi^2{}_{ijk})$ such that

$$A_i^{(n)} \subset g_n(S^1 \times \{(x,y)\ ;\ y = x \tan \frac{m\pi}{2^n} + b,\ x = 0,\ m = 1, 2, \cdots, 2^n\}).$$

And by using Theorems 10,16,17 and Theorem 14 for a suitable tubular neighborhood

$$N(A_i^{(n)}) \text{ of } A_i^{(n)} \text{ in } \mathrm{Int}H^3 - \bigcup_{i,\,j,\,k} \mathrm{Int}g_n(S^1 \times \Pi^2{}_{ijk}),$$

and by performing the same procedure as did before we may assume that each $g_n(t \times D^2)$ is perpendicular to all the annuluses $A_i^{(n)}$ along $g_n(t \times D^2) \cap A_i^{(n)}$ for all $t \in S^1$.

Now inductively we shall construct the sequences of diffeomorphisms $g_n : II^3 \longrightarrow H^3$ and Morse functions $h_n : H^3 \longrightarrow R$ which are obtained from g_n.

Now by inductive hypothesis we suppose that $g_n(t, q) = \varphi_p{}^{(n)}(t)$ for any $q \in D_0^2$ and any $t \in [0, 1-\frac{1}{2^n}]$ where $g_n(0, q) = p$. And we have

$$g_n(t, q) = g_{n-1}(t, q)$$

for all $t \in [1-\frac{1}{2^n} + \varepsilon_n, 1] \cup [0, 1-\frac{1}{2^{n-1}}]$ and all $q \in D_0^2$ where $0 < \varepsilon_n \leqq \frac{1}{3 \cdot 2^n}$.

Now for $n+1$ let $S^{(n+1)}{}_{ij0}$ be the small regions of D_0^2 that are obtained by dividing D_0^2 as above. $\{S^{(n+1)}{}_{ij0}\}$ is a refinement of $\{S^{(n)}{}_{ij0}\}$. Now by using Theorems 10, 16, 17 and theorem14 for the refinement $\{S^{(n+1)}{}_{ij0}\}$ as did before for $\{S^{(n)}{}_{ij0}\}$, as a result we have a diffeomorphism $g_{n+1} : H^3 \longrightarrow H^3$ such that each $g_{n+1}(t \times D^2)$ is perpendicular to $g_{n+1}(S^1 \times p)$ for any $p \in \partial B^2{}_{n+1m}$ and any $t \in S^1$ in H^3, and also $g_{n+1}(t \times D^2)$ is perpendicular to all the annuluses $A_i{}^{(n+1)}$. And we have

$$g_{n+1}(t, q) = g_n(t,q) \text{ for all } t \in [1-\frac{1}{2^{n+1}} + \varepsilon_{n+1}, 1] \cup [0, 1-\frac{1}{2^n}] \text{ and all } q \in D^2$$

where $0 < \varepsilon_{n+1} \leqq \frac{1}{3 \cdot 2^{n+1}}$

Now by using the Morse function h_{n+1} and the integral curves $\varphi_p{}^{(n+1)}$ of the gradient vector field grad h_{n+1} we construct a tubular neighborhood of $L = g_{n+1}(S^1 \times 0)$, that is,

$$g_{n+1} \mid [0, 1-\frac{1}{2^{n+1}} + \varepsilon_{n+1}] \times D^2 \text{ and } \{\varphi_p{}^{(n+1)}([0, 1-\frac{1}{2^{n+1}} + \varepsilon_{n+1}])\ ;\ \forall\ p \in g_{n+1}(D_0^2)\}$$

are two tubular neighborhoods of $L \cap g_{n+1}([0, 1-\frac{1}{2^{n+1}} + \varepsilon_{n+1}] \times D^2)$ in H^3. Therefore from Theorems 16, 17 these tubular neighborhoods are both equivalent in H^3, that is, with fixing all the points of ∂H^3 among the isotopy between the two tubular neighborhoods of $L \cap g_{n+1}([0, 1-\frac{1}{2^{n+1}} + \varepsilon_{n+1}] \times D^2)$.

Now by using the Extension Isotopy Theorem 14 for these tubular neighborhoods there is an enclosing ambient isotopy $f_s : H^3 \longrightarrow H^3$ such that

$$f_0 \text{ and } f_s \mid \partial H^3$$

are both identities, and

$$f_1 \circ g_{n+1} \circ x(t,q) = \varphi_p{}^{(n+1)}(t) \text{ for all } t \in [0, 1-\frac{1}{2^{n+1}}]$$

with $g_{n+1}(0, q) = p$. And since $\chi \mid ([0, 1-\frac{1}{2^n}] \times D^2)$ is an identity, we may assume also that

$$\chi \mid ([1-\frac{1}{2^{n+1}} + \varepsilon_{n+1}, 1] \times D^2)$$

is an identity, hence from the beginning, we may assume that χ is an identity. Now we replace $f_1 \circ g_{n+1} \circ \chi$ with g_{n+1}. And let h_{n+1} be the Morse function obtained from g_{n+1}.

Now from the discussions above $g_{n+1}{}^{-1}(\varphi_p{}^{(n+1)}(S^1))$ is equal to

$$S^1 \times q \text{ for } g_{n+1}(0, q) = p.$$

And thus we have the integral curves:

$$\varphi_p{}^{(n)}([0, 1-\frac{1}{2^n}]) \subset H^3 \text{ for } n = 1, 2, \cdots \text{ and all } p \in g_n(D_0^2)$$

such that $h_n \circ \varphi_p{}^{(n)}(t)) = t$ is satisfied from the preceding arguments.

Now let $\omega^{(n)}{}_{ijm}$ be the image of g_n for the small region of II^3 revised as the arguments above. And let $S^{(n)}{}_{ijm}$ and $l^{(n)}{}_{ijm}$ be the areas of base and the height respectively for the small region $\omega^{(n)}{}_{ijm}$ where $g_n{}^{-1}(l^{(n)}{}_{ijm})$ is what $g_n{}^{-1}(\varphi_p{}^{(n)}(S^1))$ is divided by the planes $z = y \tan \frac{m\pi}{2^n}$ and $y = 0$ for some $p \in H^3$. Then we have that

$$v(\omega^{(n)}{}_{ijm}) = S^{(n)}{}_{ijm} \cdot l^{(n)}{}_{ijm}.$$

Now let $L^{(n)}{}_0$ be the longitude having the minimal lenghth among all the closed ingral curves of gradh_n and let $L^{(n)}{}_{0m}$ be the divided arcs which $L^{(n)}{}_0$ is divided by

$$\{g_n(D^2{}_{nm})\ ;\ m = 0, 1, \cdots, 2^n\}.$$

And for the sake of simplicity we denote the lengths of $L^{(n)}{}_0$ and $L^{(n)}{}_{0m}$ with $L^{(n)}{}_0$ and $L^{(n)}{}_{0m}$ respectively. Then we have that

$$\Sigma S^{(n)}{}_{ijm} \cdot l^{(n)}{}_{ijm} - \Sigma S^{(n)}{}_{ijm} \cdot L^{(n)}{}_{0m} = \sum_{i,j,m} S^{(n)}{}_{ijm}(l^{(n)}{}_{ijm} - L^{(n)}{}_{0m})$$

$$\geqq \sum_{i,j} S^{(n)}{}_{ij} \sum_m (l^{(n)}{}_{ijm} - L^{(n)}{}_{0m}) \geqq \sum_{i,j} S^{(n)}{}_{ij} \ (l^{(n)}{}_{ij} - L^{(n)}{}_0\ ) \geqq 0$$

where $S^{(n)}{}_{ij} = \min\{S^{(n)}{}_{ijm}\ ;\ m = 0, 1, \cdots, 2^n\}$, $l^{(n)}{}_{ij} = \sum_m l^{(n)}{}_{ijm}$ and $L^{(n)}{}_0 = \sum_m L^{(n)}{}_{0m}$.

Now by choosing n so large for any positive number $\varepsilon > 0$, we have clearly the following inequality :

$$|\ v(H^3) - \Sigma S^{(n)}{}_{ijm} \cdot l^{(n)}{}_{ijm}\ | < \varepsilon.$$

Hence

$$\Sigma\ S^{(n)}{}_{ijm} \cdot L^{(n)}{}_{0m} \leqq \Sigma\ S^{(n)}{}_{ijm} \cdot l^{(n)}{}_{ijm} < v(H^3) + \varepsilon.$$

And let C be the infimum among the lengths of all the longitudes of H^3. Then

$$S \cdot C \leqq \sum_m S^{(n)}{}_m \cdot L^{(n)}{}_{0m} \leqq \sum_m (\sum_{i,j} S^{(n)}{}_{ijm})L^{(n)}{}_{0m}$$

$$\leqq \Sigma\ (\ \Sigma\ S^{(n)}{}_{ijm} \cdot l^{(n)}{}_{ijm}\ ) < v(H^3) + \varepsilon$$

where $S^{(n)}{}_m = \sum_{i,j} S^{(n)}{}_{ijm}$.

Thereore we have that $S \cdot C \leqq v(H^3)$. This completes the proof of Theorem 18.

Now in general let $H^3 \subset M^3$ be a handlebody of genus m. And let II_1^3 be a handlebody of genus 1 in R^3 that is the handlebody denoted by the notation II^3 in the proof of Theorem 18. Now construct a handlebody II^3 of genus m by attaching $(m\text{-}1)$-handles smoothly to II_1^3 in R^3 as illustrated in Figure 1. Let l be a longitude of II_1^3.

Now divide II^3 and II_1^3 by using the planes $z = y\tan\pi\theta$ and $y = 0$ in R^3 for any $\theta \in R$. Let $g : II^3 \longrightarrow H^3$ be a smooth diffeomorphism and the induced Rimannian metric is given in II^3 throughout g. Then let S be the infimum for all the areas of all the meridional 2-disks which are obtained by cutting II^3 with using all the planes $z = y\tan\pi\theta$ and $y = 0$ for any $\theta \in R$ and moreover have the intersection with the longitude l of II_1^3.

And let C be the infimum for all the lengths of all the longitudes of H^3 each of which is isotopic to l in H^3.

Theorem 19. *Let $H^3 \subset M^3$ be a handlebody of genus m and C, S the infima defined as above. Then $S \cdot C \leqq v(H^3)$.*

Proof. By using the same notations in the proof of Theorem 18. let H^3 be the handlebody that is obtained by rotating the 2-disk D_0^2 around x-axis in R^3. And also we denote the handlebody divided as in the proof of Theorem 18 with H_1^3.

Now construct a handlebody H^3 of genus m by attaching $(m-1)$-handles smoothly to H_1^3 in R^3 as illustrated in Figure 1.

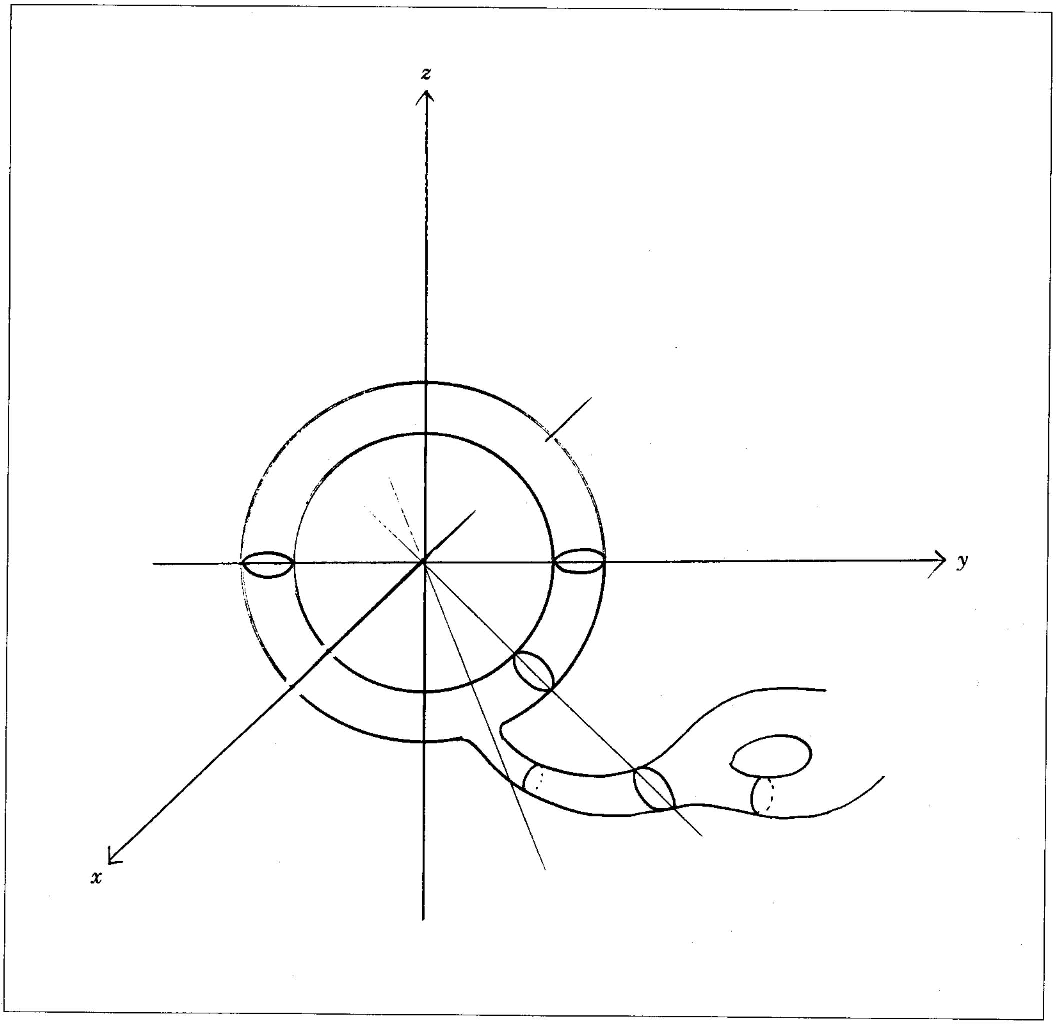

Figure 1

Now let $g : II^3 \longrightarrow H^3$ be a diffeomorphism such that $g \mid II_1^3$ is what is used as II^3 in the proof of Theorem18. Since II_1^3 is divided as in the proof of Theorem 18, therefore there are a sequence of $(g \mid II_1^3)_n$ and Morse functions which are obtained from $(g \mid II_1^3)_n$. Now we have a sequence of diffeomorphisms $g_n : II^3 \longrightarrow H^3$ by extending $(g \mid II_1^3)_n$ onto II^3. Now let h_n be Morse functions obtained from g_n for all n such that all the integral curves of grad h_n are perpendicular to all the surfaces

$$g_n(\{z = y \tan\pi\theta\} \cap II^3) \text{ and } g_n(\{(x, 0, z)) ; x \in R^1, z \in R^1\} \cap II^3)$$

outside $g_n(II_1^3)$.

Now divide II^3 by using all the palnes $z = y \tan\pi \theta, y = 0$ and moreover divide H^3 by using the surfaces $g_n(\{z = y \tan\pi\theta\} \cap II^3)$, $g_n(\{(x, 0, z) ; x \in R^1, z \in R^1\} \cap II^3)$ suitably and the Morse functions h_n. Now let $\omega^{(n)}{}_{ijm}$ be the small regions that are obtained by dividing H^3 thus. And by using all the regions $\omega^{(n)}{}_{ijm}$ that have the intersection with all the meridional 2-disks which are obtained by cutting II^3 with using all the planes

$$g_n(\{z = y \tan\pi \theta\} \cap II^3) \text{ and } g_n(\{(x, 0, z) ; x \in R^1, z \in R^1\} \cap II^3)$$

and moreover have the intersection with the longitude l of II_1^3 then we can prove Theorem 19 as the same way as the proof of Theorem18 hereafter.

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