ANALYSIS OF THE Z TERM IN THE VARIATION OF LATITUDE (I)

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緯度変化における Z 項の解析 (I)

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1. Introduction

In 1902, Dr. H. KIMURA modified the equation of Pole's motion as follows:

$$\Delta\varphi = x\cos\lambda + y\sin\lambda + z \qquad (1.1)$$

and minimized the errors.

Despite many scholar's efforts in these 65 years since then, the Z Term has not yet been solved.

Therefore, to reconsider its foundation and to get a solution, I shall investigate the Z term mathematically, and get the modified value of variation of latitude $\Delta\varphi'$:

$$\Delta\varphi' = x\cos\lambda + y\sin\lambda \qquad (1.2)$$

From (1.2), presuming the differences between observed values $\Delta\varphi'$ and calculated ones $\Delta\phi$ as new Z, the new Z is the local errors.

2. Notation of the Most Probable Values

Linear equation of unknown values x and y will be represented as

$$ax + by = c \qquad (2.1)$$

By the least square method of the data of n observations, we can get the most probable values X and Y respectively.

So, (2.1) may be described as

$$L\ (ax + by = c) = (X,\ Y) \qquad (2.2)$$

In the same manner, in the following equation

$$ax + by + z = c \qquad (2.3)$$

let the most probable values be represented as X_0, Y_0 and Z_0, and we get

$$L\ (ax + by + z = c) = (X_0, Y_0, Z_0) \qquad (2.4)$$

where z is of course not a erroneous term.

Now, in (2.2) and (2.4), changing c term to $c+u$, we can get the following results, by solving each of the normal equations.

(1) Putting $L\ (ax+by=c)=(X,\ Y)$ then, we get

$$L\,(ax+by=c+u) = (X+\xi,\ Y+\eta)$$

where ξ and η are the functions of u.

(2) And also, let

$L\ (ax+by+z=c)=(X_0,\ Y_0,\ Z_0)$,

then $L(ax+by+z=c+u)=(X_0,\ Y_0,\ Z_0+u)$

Corollary 1. $L\ (ax+by+z=c\text{-}z_0)=(X_0,\ Y_0,\ 0)$

This equation should be called basic solution, and is denoted as $L\ (ax+by=c\text{-}z_0)=(X_0,\ Y_0)$

Conseqently, the following corollaries are obtained.

Corollary 2. $L(ax+by=c\text{-}z_0) \neq L(ax+by=c)$

Corollary 3. $L(ax+by+z=c\text{-}\bar{c})=(X_0,\ Y_0,\ Z_0\text{-}\bar{c})$

Corollary 4. $L(ax+by=c\text{-}z_0)=L(ax+by=c'\text{-}z_0')$

3. Errors and the z_0 value

Let us put the determinants of coefficients of normal equations $L\ (ax+by=c)$ and $L\ (ax+by+z=c)$ as follows:

$$D=\begin{vmatrix}[aa] & [ab] & [ac]\\ [ba] & [bb] & [bc]\\ [ca] & [cb] & [cc]\end{vmatrix},\quad \Delta=\begin{vmatrix}[aa] & [ab] & [ac]\\ [ba] & [bb] & [bc]\\ [a] & [b] & [c]\end{vmatrix}$$

$$D_1=\begin{vmatrix}[aa] & [ab] & [a]\\ [ba] & [bb] & [b]\\ [a] & [b] & n\end{vmatrix},\quad D_2=\begin{vmatrix}[aa] & [ab]\\ [ba] & [bb]\end{vmatrix}$$

Then, from these determinants, the following results are secured.

(3) The value of Z_0 of the equation $L(ax+by+z=c)=(X_0,\ Y_0,\ Z_0)$ is

$$z_0 = \frac{\Delta}{D_1} \qquad (3.1)$$

(4) Put $\bar{\varepsilon}$ as a mean of errors of the equation $L(ax+by=c)=(X,\ Y)$, and we get

then

$$\bar{\varepsilon} = \frac{\Delta}{n D_2},\quad \sigma^2 = \frac{1}{n-2}\cdot\frac{D}{D_1} \qquad (3.2)$$

Corollary

$$z_0 = \frac{n D_2}{D_1}\cdot\bar{\varepsilon} \qquad (3.3)$$

(5) Then, the mean of errors of $L\ (ax+by+z=c)=(X_0,\ Y_0,\ Z_0)$ is zero.

4. Partial least square method

When we denote the error of the equation:

$L(ax+by=c)=(X, Y)$ (4.1)

as ξ, and suppose that ε contains both common ζ and accidental η errors, where ζ is an unknown value, so we can solve by the following form.

$L\ (ax+by+\zeta=c)=(X_0,\ Y_0,\ Z_0')$ (4.2)

At the same time, by (4.2), we can merely minimize the sum of square of only a part η of errors.

So, we may call this method as the partial least square method.

The root of this equation is of coruse not a root of (4.1) itself. Accordingly, I call the former as affine-root and Z'_0 as error-adjusting term.

(6) By changing the equation $L\ (ax+by=c)=(X,Y)$ to $L(ax+by+z=c)=(X_0,\ Y_0,\ Z_0)$ the mean of errors will be adjusted to zero by Z_0, but its roots are merely affine-roots.

This fact is very inportant, that is, despite the fact that between x and y, there is $ax+by=c$, if we change it unreasonably to $ax+by+z=c$ and solve the equation, then, the roots obtained will be affine-ones.

Now, such a instance will be cited by the following.

Example:

Let there be lots of apples and oranges as follows:

A.	One apple and one orange	price	20	yen
B.	Two apples and three oranges	〃	40	〃
C.	Three apples and two oranges	〃	50	〃
D.	Four apples and two oranges	〃	60	〃

Find the most probable value of each price of an apple and orange.

Solution:

Let the price of an apple and an orange be X and Y respectively, and presume that there are no other factors affecting the price.

Then, by solving the equation $L\,(ax+by=c)=(X,\ Y)$,

we get 12.7 and 5.2 respectively.

On the other hand, the value of X_0, Y_0 and Z_0 may be 12.2, 3.9 and 4.2 respectively, by solving the equation $L(ax+by+z+c=)=(X_0,\ Y_0,\ Z_0)$ formally. Here, what is the meaning of $Z_0=4.2$?

Of course, this Z_0 is an error-adjusting term, and so it will be helpful to estimate the prices of each lot separately, but unavailable to the estimation of the most probable values.

We should think that the problems of Z term are involved in this fact.

5. *Z* term in the variation of latitude

The equation of Pole's motion at a certain station $(\lambda_i,\ \varphi_i)$ is given as

$$\Delta\varphi_i = x\cos\lambda_i + y\sin\lambda_i \qquad (5.1)$$

Here $\Delta\varphi$ is decided by the equation

$$\Delta\varphi_i=\frac{1}{2}\,(\delta_s+\delta_n)+\frac{1}{2}\,(Z_s-Z_n)-\varphi_i \qquad (5.2)$$

Let us suppose that there are declination variation $\Delta\delta$ and adjusted latitude variation $\Delta\varphi'$, so (5.1) will be described as follows:

$$\Delta\varphi_i' = x\cos\lambda_i + y\sin\lambda_i\,, \qquad (5.3)$$

where $\Delta\varphi_i' = \Delta\varphi_i - \Delta\delta$.

Therefore, the most probable solution can be obtained by the equation

$$L\,(x\cos\lambda + y\sin\lambda = \Delta\varphi') = (X,\ Y) \qquad (5.4)$$

Supposing that the error contains both common error ζ and accidental error η, then formally, it will be solved by means of the equation

$$L\ (x\cos\lambda+y\sin\lambda+\zeta=\Delta\varphi')=(X_0,\ Y_0,\ Z_0) \qquad (5.5)$$

From the results of (6), the roots of (5.5) will be affine-roots and Z_0 will be an adjusted error.

(7) Let the adjusted value of variation of declination be given as $\Delta\varphi'$ the most probable value of pole $(x,\ y)$ will be obtained as the roots of

$$L\ (x\cos\lambda + y\sin\lambda=\Delta\varphi')=(X,\ Y)$$

and the roots of $L\ (x\cos\lambda\ +y\sin\lambda\ +z=\Delta\varphi')=(X_0,\ Y_0,\ Z_0)$ are affine-roots and Z_0 are adjusted errors.

In these equations, $\Delta\delta$ means the variation of declination of j-group stars to be observed at t time, and $\Delta\delta$ is common at each observatory.

Accordingly, quantities of adjusted latitude variation $\Delta\varphi'$ at i-station will be calculated as follows:

$$\Delta\varphi_i' = \Delta\varphi_i - \Delta\delta\ . \qquad (5.6)$$

Let the $\Delta\varphi_i(j,\ e)$ be the observed value of $(j+1)$ group stars in the afternoon of t-day at i-station, and we get $\Delta\varphi_i(j,\ m)$ as value of j-group stars observed in the next morning.

Then, there will be given two equations as follows:

$$\Delta\varphi_i(j,e)=\Delta\varphi_i'(j)+\Delta\delta(j+1)$$
$$\Delta\varphi_i(j,m)=\Delta\varphi_i'(j)+\Delta\delta(j)\ .$$

From these, the equation

$$\Delta\varphi_i(j,e)-\Delta\varphi_i(j,m)=\Delta\delta(j+1)-\Delta\delta(j) \qquad (5.7)$$

will be led.

By averaging the values at n observatories,

$$\Delta\delta(j+1)-\Delta\delta(j)=\frac{1}{n}\sum_{i=1}^{n}\left\{\Delta\varphi_i(j,l)-\Delta\varphi_i(j,m)\right\} \qquad (5.8)$$

and to simplify the above equation, put the right region as $d(j)$ and then adjust the (5.8) by summing up with j.

so
$$\Delta\delta(j+1)=\Delta\delta(j)+\sum_{j=1}^{j}d(j) \qquad (5.9)$$

Accordingly, the value of $d(j)$ will be checked by the equations (5.7), (5.8) or (5.9), and moreover, the order of errors will also be obtained.

6. Conclusion

Let $L\ (x\cos+y\sin\lambda+\Delta\varphi')=(X,Y)$ be given then the value of variation of Latitude $\Delta\phi_i$ at i-station, will be calculated by the following equation.

$$\Delta\phi_i=X\cos\lambda_i+Y\sin\lambda_i \qquad (6.1)$$

In (6.1), let Z_i' represent differences between calculated $\Delta\phi_i$ and observed value $\Delta\varphi_i$ so

therefore, $$\Delta\varphi_i'=X\cos\lambda_i+Y\sin\lambda_i+Z_i \qquad (6.2)$$

Here, Z contains variation caused by i-station of observatory and another errors, and so, it might be possible to analyse them at each station, separately mixed with other factors, but they can not be obtained only by the analysis of the equation (6.2).

Then we have obtained the following results.

1. The roots of the equation $L\ (x\cos\lambda+y\sin\lambda+z=\Delta\varphi)=(X_0, Y_0, Z_0)$ are affine-roots, Z_0 is an adjusted error, and the mean of errors will be zero by adding Z_0.
2. The roots of the $L\ (x\cos\lambda+y\sin\lambda)=(X,Y)$ are the most optimum values,
   where $\Delta\varphi'=\Delta\varphi-\Delta\delta$
3. Let the difference between the calculated value $\Delta\phi$ and the observed value $\Delta\varphi'$ be put as Z,
   where $\Delta\varphi=X\cos\lambda+Y\sin\lambda$
   Then $Z=\Delta\varphi'-\Delta\phi$
   and $\Delta\varphi'=X\cos\lambda+Y\sin\lambda+Z$

The Z contains both variation and errors of the observatory, and so can not be analysed without other factors.

要　　約

1902年木村栄博士が，極の軌道方程式を

$$\Delta\varphi=x\cos\lambda+y\sin\lambda+z \qquad (1.1)$$

と修正し，誤差を小さくしたが，その後65年間の多くの人の研究にもかかわらず，この z はいまだに解明されていない。ここに(1.1)式の立式を反省し，z 項を数学的に吟味し，緯度変化量の修正値 $\Delta\varphi'$ を求め，しかる後に z を挿入すると x，y の値が歪む。よって極の新しい軌道方程式として z を入れない次式

$$\Delta\varphi'=x\cos\lambda+y\sin\lambda \qquad (1.2)$$

を提案する。

ここに観測値 $\Delta\varphi'$ と計算値 $\Delta\phi$ との差を新しい z とすると，これは地方誤差になっている。したがって各地においてそれぞれの z の研究が必要となる。 (1967. 3. 3)