ON A LEFT HILBERT ALGEBRA

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Let A be a left Hilbert algebra with the $\sharp$-involution and H be its completion. Put, for each $x \in A$, $\pi_0(x)y = xy$, $y \in A$, and we see that $\pi_0(x)$ can be extended to the bounded operator $\pi(x)$ on H. Because of the density of A^2 in H, we see that $\pi(A)$ is the non-degenerate *-algebra on H. Note that $\pi(xy) = \pi(x)\pi(y)$, $\pi(x)^* = \pi(x^\sharp)$, $x, y \in A$. We denote the weakly closure of $\pi(A)$ in $B(H)$ by $L(A)$ and we call it the left von Neumann algebra associated to A.

Now, let S be the minimal extension of the pre-closed conjugate linear operator; $x \longmapsto x^\sharp$, $x \in A$. Let $S = J\Delta^{1/2}$ be the polar decomposition of S where $\Delta = S^*S$. We put $F = S^*$ and for each element y of the domain $D(F)$ of F we define $\pi'_0(y)x = \pi(x)y$, $x \in A$. Then $\pi'_0(y)$ is a densely defined pre-closed operator on H and it can be extended to the closed operator $\pi'(y)$ on H. Put $A' = \{y \in D(F);\ \pi'(y) \text{ is bounded}\}$. Then A' is the right Hilbert algebra with the product $xy = \pi'(y)x$ and the involution $x^\flat = Fx$, $x, y \in A'$.

Similarly, we define π'', A''. Then we easily see that $A \subset A''$ and $\pi''(x) = \pi(x)$, $x \in A$. We may denote π'' by π without confusion.

In this paper we shall discuss the characterization of a Tomita algebra with respect to the relation between a left Hilbert algebra and the right Hilbert algebra, and its application to a quasi-unitary algebra. Some results in this paper are known. We shall completely give the proof.

Proposition 1. *Let A_1 and A_2 be two left Hilbert algebras. Suppose that A_1 is the left Hilbert subalgebra of A_2 and the completion of A_1 is equal to that of A_2. Then A_2' is the right Hilbert subalgebra of A'_1.*

Proof. Let S_1 and S_2 be the involution of A_1 and A_2 respectively. We denote the notation π of A_1 and A_2 by π_1 and π_2 respectively. We have $S_1 \subset S_2$ and $\pi_1(x) = \pi_2(x)$, $x \in A_1$. Then, for $y \in A'_2$, we obtain

$$\pi_1'(y)x = \pi_1(x)y = \pi_2(x)y = \pi_2'(y)x, \quad x \in A_1.$$

Since $\pi_2'(y)$ is bounded, $\pi_1'(y)$ is also bounded. This implies that y falls in A_1'.
Q.E.D.

Proposition 2. i) *If A is a Tomita algebra, then $A \subset A'$.*

ii) *Let A be an achieved left Hilbert algebra and let A_0 be the maximal Tomita algebra of A. If $S = F$ on A_0, then $A = A' = A_0$ and A is a unimodular Hilbert algebra.*

Proof. i) Let S be the involution of A and let $\Delta(\alpha)$ be the modular automorphism group of A. We have $A \subset D(F)$ because of $F \supset \Delta(1) S$. For $x, y \in A$, we have

$$\|\pi'(x) y\| = \|yx\| = \|J(yx)\| = \|(Jx)(Jy)\| = \|\pi(Jx) Jy\|$$
$$\leqq \|\pi(Jx)\| \, \|Jy\| = \|\pi(Jx)\| \, \|y\|,$$

where J is the canonical conjugation of A. Hence $\pi'(x)$ is bounded, that is, an element x belongs to A'. Consequently we have $A \subset A'$.

ii) Note that $A_0 = \{x \in A \cap \bigcap_{\alpha \in C} D(\Delta^\alpha),\ \Delta^\alpha x \in A,\ \alpha \in C\}$.

We have

$$\Delta x = FSx = S^2 x = x, \ x \in A_0.$$

Since A_0 is a core of Δ, we have $\Delta = 1$. It is clear that $A = A' = A_0$. Q.E.D.

Remark Note in [Prop. 1] that $A_1 \subset A_2$ as a set for the left or right Hilbert algebras A_1, A_2, does not always imply $A_1' \supset A_2'$. For example, we have $A_0 \subset A'$ where A_0 is a maximal Tomita algebra in the achieved left Hilbert algebra A. (see. [4] Lemma 5. 4. ii)) However we cannot expect that this leads us to $A_0' \supset A''$ $= A$ and $A = A_0'' \subset A''' = A'$.

Theorem 1. *Let A be an achieved left Hilbert algebra. Then the following two conditions are equivalent.*

i) $A = A'$.

ii) *A is a Tomita algebra.*

In particular, if we put $A = M\xi_0$ for a von Neumann algebra M with a cyclic and separating vector ξ_0, then the avove each condition implies that M is of type finite.

Proof. i)⇒ii) Let Δ be the modular operator of A. We get $\Delta^{1/2} A = A' = A$. (see. [3] Cor. 10. 1) It follows that $\Delta^\alpha A = A$, $\alpha \in C$.

In fact, we easily see that $\Delta^{n/2} A = A$, $n = \pm 1, \pm 2, \cdots$, so that $A \subset D(\Delta^\alpha)$, $\alpha \in C$ and $\Delta^\alpha x \in D(\Delta^{1/2})$, $x \in A$, $\alpha \in C$. Now we shall prove $\pi(\Delta^\alpha x)$, $x \in A$, $\alpha \in C$, is bounded. For a fixed positive integer m_0 and $x \in A$, $y, z \in A'$, we define

$$f(\alpha) = (\pi(\Delta^\alpha x) y, z) = (\Delta^\alpha x, z y^\flat), \ \alpha \in C, \ 0 \leqq \mathrm{Re}\ \alpha \leqq m_0.$$

Then $f(\alpha)$ is continuous on and analytic in the strip $0 \leqq \mathrm{Re}\ \alpha \leqq m_0$. We obtain for $t \in R$,

$$\begin{aligned} |f(it)| &= |(\pi(\Delta^{ti} x) y, z)| \\ &= |(\Delta^{it} \pi(x) \Delta^{-it} y, z)| \\ &\leqq \|\Delta^{it} \pi(x) \Delta^{-it}\| \, \|y\| \, \|z\| \\ &\leqq \|\pi(x)\| \, \|y\| \, \|z\|, \end{aligned}$$

and

$$\begin{aligned}|f(m_0+it)| &= |(\pi(\Delta^{m_0+it}x)y, z)| \\ &= |(\Delta^{it}\pi(\Delta^{m_0}x)\Delta^{-it}y, z)| \\ &\leqq \|\pi(\Delta^{m_0}x)\| \, \|y\| \, \|z\| .\end{aligned}$$

Note that $\Delta^{m_0}x \in A$ and then $\pi(\Delta^{m_0}x)$ is bounded. By Phragmen-Lindelöf theorem, we have for $0 \leqq s \leqq m_0$,

$$|f(s+it)|^{m_0} \leqq (\|\pi(x)\| \, \|y\| \, \|z\|)^{m_0-s} \times (\|\pi(\Delta^{m_0}x)\| \, \|y\| \, \|z\|)^s.$$

Hence we have

$$\|\pi(\Delta^{s+it}x)y\| \leqq \|\pi(x)\|^{1-s/m_0} \|\pi(\Delta^{m_0}x)\|^{s/m_0} \|y\|.$$

This inequality holds for all $y \in A'$, so that $\pi(\Delta^\alpha x)$ is bounded, that is, $\Delta^\alpha x$ falls in A for $x \in A$ and $0 \leqq \mathrm{Re}\ \alpha \leqq m_0$. Similarly, we get $\Delta^\alpha x \in A$, $x \in A$ for $-m_0 \leqq \mathrm{Re}\ \alpha \leqq m_0$. It follows that $\Delta^\alpha x \in A$, $x \in A$, $\alpha \in C$. Then the maximal Tomita algebra A_0 in A is equal to A, so that A is a Tomita algebra.

ii)⇒i) If A is a Tomita algebra, then $A \subset A'$ by [Prop. 2. i)]. Since

$$A' = JA \subset JA' = A$$

by ([3] Cor. 10. 1) where J is the canonical involution of A, it follows that $A = A'$. Therefore ii) implies i).

Let M be a von Neumann algebra on a Hilbert space H and ξ_0 be the cyclic and separating vector in H for M. We can consider $M\xi_0$ as a left Hilbert algebra A with the product: $(x\xi_0)(y\xi_0) = xy\xi_0$, $x, y \in M$ and the involution: $(x\xi_0)^\# = x^*\xi_0$, $x \in M$. Then we see $A' = M'\xi_0$. (see. [4]) Consequently $A = M\xi_0$ is the achieved left Hilbert algebra. If $M\xi_0 = M'\xi_0$, then M is finite. (see. [2]) Q.E.D.

Remark If an achieved left Hilbert algebra A is uni-modular, then we easily see that $A = A'$. And if a left Hilbert algebra A is finite dimentional, then it is evident that $A = A'$.

A simple example shows that there exists a non-unimodular left Hilbert algebra A which is equal to the right Hilbert algebra A' as a set.

Let H_2 be a two-dimensional Hilbert space. We consider the von Neumann algebra $M = B(H_2) \otimes I_{H_2}$ on the Hilbert space $H_2 \otimes H_2$ where $B(H_2)$ is the algebra of all bounded operators on H_2 and I_{h_2} is the algebra of identity operators on H_2. Let e_1, e_2 be the orthonormal basis of H_2 and we put

$$\xi_0 = 2e_1 \otimes e_1 + e_2 \otimes e_2 ,$$

then we see that ξ_0 is a cyclic and separating vector for M without difficulty. We note that $M' = I_{H_2} \otimes B(H_2)$. Now we put

$$x_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in B(H_2).$$

Then we have

$$\begin{aligned}\|(x_1\otimes 1)\xi_0\|^2 &= \|2x_1e_1\otimes e_1+x_1e_2\otimes e_2\|^2\\ &=4\ \|x_1e_1\|^2+\|x_1e_2\|^2\\ &=4\left\|\begin{pmatrix}1\\0\end{pmatrix}\right\|^2+\left\|\begin{pmatrix}1\\0\end{pmatrix}\right\|^2\\ &=5\end{aligned}$$

and

$$\begin{aligned}\|(x_1\otimes 1)^*\xi_0\|^2 &= \|2x_1^*e_1\otimes e_1+x_1^*e_2\otimes e_2\|^2\\ &=4\ \|x_1^*e_1\|^2+\|x_1^*e_2\|^2\\ &=4\left\|\begin{pmatrix}1\\1\end{pmatrix}\right\|^2+\left\|\begin{pmatrix}0\\0\end{pmatrix}\right\|^2\\ &=8,\end{aligned}$$

so that $\|((x_1\otimes 1)\xi_0))^\#\|=\|(x_1\otimes 1)^*\xi_0\|\neq\|(x_1\otimes 1)\xi_0\|$. Hence the $\#$-involution of A is not isometric and therefore A is not uni-modular.

Let M be a von Neumann algebra with a cyclic and separating vector. The author proved in his work [2] that M is finite if and only if M has the property (J), that is, there exists a cyclic and separating vector ξ_0 such that $M\xi_0=M'\xi_0$. We immediately see that $\pi(x\xi_0)=x$ for any element x of M, so that the left von Neumann algebra $L(M\xi_0)$ associated to the left Hilbert algebra $M\xi_0$ is equal to M. It may be interesting to characterize the type of the left von Neumann algebra $L(A)$ associated to a general achieved left Hilbert algebra A which is equal to the right Hilbert algebra A' under a certain condition.

Theorem 2. i) *A quasi-unitary algebra can be a left Hilbert algebra. Furthermore if the quasi-unitary algebra is achieved in the sense of a left Hilbert algebra, then it is a Tomita algebra.*

ii) *Conversely a Tomita algebra can be a quasi-unitary algebra.*

J. Dixmer defined the quasi-unitary algebra in [1]. If A is the algebra over C with the pre-Hilbert structure (x, y) and satisfies the following conditions;

1) $x\longmapsto x^\wedge$ is the automorphism on A,

2) A has the involution: $x\longmapsto x^*$ such that

(i) $(x, x^\wedge)\geqq 0,\ x\in A$

(ii) $(xy, z)=(y, x^{*\wedge}z)$,

(iii) $(x, x)=(x^*, x^*)$,

(iv) $y\longmapsto xy$ is continuous,

(v) The set of all elements $xy+(xy)^\wedge,\ x, y\in A$, is dense in A,

then A is called the quasi-unitary algebra.

Proof. i) Suppose that A is a quasi-unitary algebra. We put

$$Sx=x^{*\wedge},\ x\in A.$$

Then S is the pre-closed involution of A. In fact, if we put

$$Px=x^*, \qquad Qx=x^\wedge, \qquad x\in A,$$

then $S=QP$ and P is isometric and $Q\geqq 0$ by condition 2). It follows that

$$P^{**}=P^*,\ Q\subset Q^*.$$

Since $D(QP)=A$, QP is densely defined on the completion H of A. We have

$$(QP)^*\supset P^*Q^*\supset PQ.$$

Hence $D((QP)^*)\supset A$ and then $S=QP$ is pre-closed. Moreover we have

$$(QP)^{**}\subset (P^*Q^*)^*=Q^{**}P^{**}$$

because of the boundedness of P^*. Now for a given $x\in D(Q^{**}P^{**})$, we have $P^{**}x\in D(Q^{**})$. Condition 2) (v) shows that A^2 is a core set of Q^{**}. (cf. [3] Lemma 1.1) Therefore A is core of Q^{**}. Hence there exists a sequence $\{x_n\}$ in A such that

$$\lim x_n=P^{**}x,$$
$$\lim Q^{**}x_n=Q^{**}P^{**}x.$$

Thus $P^{**}x_n\longrightarrow x$, so that we have

$$\lim Px_n=x,$$
$$(QP)^{**}Px_n=QP^2x_n=Qx_n.$$

Since $(QP)^{**}$ is closed and the sequence $\{Px_n\}$ is convergent, it follows that

$$x\in D((QP)^{**}),$$
$$(QP)^{**}x=\lim (QP)^{**}Px_n=\lim Qx_n=Q^{**}P^{**}x.$$

Consequently we obtain

$$(QP)^{**}=Q^{**}P^{**}.$$

Consider the right polar decomposition of the closure S^{**} of $S=QP$

$$S^{**}=\Delta^{-1/2}J,\ (\Delta=S^*S^{**}),$$

and we see that

$$\Delta^{-1/2}=Q^{**},\ J=P^{**}$$

because of the unicity of the polar decomposition. Therefore

$$\Delta^{-1/2}A=Q^{**}A=A.$$

Then we have

$$\Delta^{1/2}A=A. \tag{*}$$

After all, we can consider A as the left Hilbert algebra with the involution S, and if A is achieved in the sense of the left Hilbert algebra, then A can be a Tomita algebra because of $A=A'$ by the equality (*) and by using [Theorem 1].

ii) Suppose that A is a Tomita algebra. Let Δ and J be the modular operator and the canonical conjugation of A respectively. We define

$$x^{\wedge}=\Delta^{-1/2}x,\ x^{*}=Jx,\ x\in A.$$

Note that condition 2) (v) is fulfilled because A^2 is a core set of $\Delta^{-1/2}$, and we easily see that A can be a quasi-unitary algebra. This completes the proof. Q.E.D.

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