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DECOMPOSITION OF AN ORDER ISOMORPHISM BETWEEN MATRIX-ORDERED HILBERT SPACES

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ABSTRACT. The purpose of this note is to show that any order isomorphism between noncommutative L^2-spaces associated with von Neumann algebras is decomposed into a sum of a completely positive map and a completely co-positive map. The result is an L^2 version of a theorem of Kadison for a Jordan isomorphism on operator algebras.

1. INTRODUCTION

In the theory of operator algebras, a notion of self-dual cones was studied by A. Connes [1], and he characterized a standard Hilbert space. In [9] L. M. Schmitt and G. Wittstock introduced a matrix-ordered Hilbert space to handle a noncommutative order and characterized it using the face property of the family of self-dual cones. From the point of view of the complete positivity of the maps, we shall consider a decomposition theorem of an order isomorphism on matrix-ordered Hilbert spaces.

Let $\mathcal{H}$ be a Hilbert space over a complex number field $\mathbb{C}$, and let $\mathcal{H}^+$ be a self-dual cone in $\mathcal{H}$. A set of all $n \times n$ matrices is denoted by M_n. Put $\mathcal{H}_n = \mathcal{H} \otimes M_n (= M_n(\mathcal{H}))$ for $n \in \mathbb{N}$. Suppose that $(\mathcal{H}, \mathcal{H}_n^+, n \in \mathbb{N})$ and $(\tilde{\mathcal{H}}, \tilde{\mathcal{H}}_n, n \in \mathbb{N})$ are matrix-ordered Hilbert spaces. A linear map A of $\mathcal{H}$ into $\tilde{\mathcal{H}}$ is said to be n-positive (resp. n-co-positive) when the multiplicity map $A_n (= A \otimes \mathrm{id}_n)$ satisfies $A_n\mathcal{H}_n^+ \subset \tilde{\mathcal{H}}_n^+$ (resp. ${}^t(A_n\mathcal{H}_n^+) \subset \tilde{\mathcal{H}}_n^+$). Here ${}^t(\cdot)$ denotes a set of all transposed matrices. When A is n-positive (resp. n-co-positive) for all $n \in \mathbb{N}$, A is said to be completely positive (resp. completely co-positive). We refer mainly to [11] for standard results in the theory of operator algebras. We use the notation as introduced in [9] with respect to matrix-ordered standard forms.

2. RESULTS

We first generalize a theorem of A. Connes [1] for the polar decomposition of an order isomorphism to the case where a von Neumann algebra is non-σ-finite.

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Proposition 1. *Let* $(\mathcal{M}, \mathcal{H}, J, \mathcal{H}^+)$ *and* $(\tilde{\mathcal{M}}, \tilde{\mathcal{H}}, \tilde{J}, \tilde{\mathcal{H}}^+)$ *be standard forms, and let* A *be a linear bijection of* $\mathcal{H}$ *onto* $\tilde{\mathcal{H}}$ *satisfying* $A\mathcal{H}^+ = \tilde{\mathcal{H}}^+$. *Then for a polar decomposition* $A = U|A|$ *of* A *we obtain the following properties:*

(1) *There exists a unique invertible operator* B *in* $\mathcal{M}^+$ *such that* $|A| = BJBJ$ (cf. [4, Corollary II.3.2]).

(2) *There exists a unique Jordan *-isomorphism* α *of* $\mathcal{M}$ *onto* $\tilde{\mathcal{M}}$ *such that*

$$(\alpha(X)\xi, \xi) = (XU^{-1}\xi, U^{-1}\xi)$$

for all $X \in \mathcal{M}, \xi \in \tilde{\mathcal{H}}^+$.

Proof. (1) Let $\mathcal{M}$ be non-σ-finite. Choose an increasing net $\{p_i\}_{i\in I}$ of σ-finite projections in $\mathcal{M}$ converging strongly to 1. Put $q_i = p_iJp_iJ$. By [1, Theorem 4.2] $q_i\mathcal{H}^+$ is a closed face of $\tilde{\mathcal{H}}^+$. Since A is an order isomorphism, $A(q_i\mathcal{H}^+)$ is a closed face of $\tilde{\mathcal{H}}^+$. There then exists a σ-finite projection $p_i' \in \tilde{\mathcal{M}}$ such that $A(q_i\mathcal{H}^+) = q_i'\tilde{\mathcal{H}}^+$ where q_i' denotes $p_i'Jp_i'J$. Hence $q_i'Aq_i$ is an order isomorphism of $q_i\mathcal{H}^+$ onto $q_i'\tilde{\mathcal{H}}^+$. These cones appear respectively in the reduced standard forms $(q_i\mathcal{M}q_i, q_i\mathcal{H}, q_iJq_i, q_i\mathcal{H}^+)$ and $(q_i'\tilde{\mathcal{M}}q_i', q_i'\tilde{\mathcal{H}}, q_i'Jq_i', q_i'\tilde{\mathcal{H}}^+)$. Put $A_i = (q_i'Aq_i)^*q_i'Aq_i$. Then $A_i \in q_i\mathcal{M}^+q_i$ is an order automorphism on $q_i\mathcal{H}^+$. By [3, Theorem 3.3] there exists a unique invertible operator $B_i \in q_i\mathcal{M}^+q_i$ such that $A_i = B_iJ_iB_iJ_i$, where J_i denotes q_iJq_i. Taking a logarithm of both sides, we have $\log A_i = \log B_i + J_i(\log_i B_i)J_i$. Since $\{A_i\}$ is bounded, $\{\log B_i\}$ is bounded. Indeed, we have in a standard form that a map

$$X \mapsto \delta_X = \tfrac{1}{2}(X + JXJ)$$

is a Jordan isomorphism of a selfadjoint part of $\mathcal{M}$ into a selfadjoint part of a set of all order derivations $D(\mathcal{H}^+)$ by [4, Corollary VI.2.3]. It is known that any isomorphism of a JB-algebra into another JB-algebra is an isometry (see [3, Proposition 3,4.3]). Hence

$$\| \delta_X \| = \| X \|, \quad X \in \mathcal{M}_{s.a.}.$$

Thus $\{\log B_i\}$ is bounded. It follows that $\{p_iB_ip_i\}$ is bounded because $p_i\mathcal{M}p_i$ and $q_i\mathcal{M}q_i$ are *-isomorphic. Therefore, one can find a subnet of $\{p_i \log B_ip_i\}$ that converges to some element $C \in \mathcal{M}^+$ in the σ-weak topology. We may index the subnet as the same $i \in I$. We then have for $\xi, \eta \in \mathcal{H}$,

$$\begin{aligned}((C + JCJ)q_j\xi, q_j\eta) &= \lim_i((p_i(\log B_i)p_i + Jp_i(\log B_i)p_iJ)q_j\xi, q_j\eta) \\ &= ((\log B_j + J_j(\log B_j)J_j)q_j\xi, q_j\eta) \\ &= \lim_i(\log A_iq_j\xi, q_j\eta) \\ &= (\log A^*Aq_j\xi, q_j\eta),\end{aligned}$$

using the facts that $q_iXq_iJq_iXq_iJq_i = p_iXp_iJp_iXp_iJq_i$ for all $X \in \mathcal{M}$, and under the strong topology $\{A_i\}$ converges to A^*A; hence $\{q_i(\log A_i)q_i\}$ converges to $\log A^*A$. Since $\bigcup_{i\in I} q_i\mathcal{H}$ is dense in $\mathcal{H}$, we obtain the equality $C + JCJ = \log A^*A$. Therefore, $e^CJe^CJ = A^*A$. Thus there exists an element $B \in \mathcal{M}^+$ such that $|A| = BJBJ$. Since, in addition, $q_iBq_iJq_iBq_iJq_i = q_i|A|q_i$, one easily sees the invertibility and the unicity of B using the same properties as in the σ-finite case.

(2) From (1) we have $U = AB^{-1}JB^{-1}J$. It follows that U is an isometry satisfying $U\mathcal{H}^+ = \tilde{\mathcal{H}}^+$. Let p_i and q_i be as in (1). There then exists a σ-finite projection $p_i' \in \tilde{\mathcal{M}}$ such that $U(q_i\mathcal{H}^+) = q_i'\tilde{\mathcal{H}}^+$ with $q_i' = p_i'Jp_i'J$. Using also [1,

Theorem 3.3], one can find a unique Jordan $*$-isomorphism α_i of $q_i\mathcal{M}q_i$ onto $q_i'\tilde{\mathcal{M}}q_i'$ such that

$$(\alpha_i(q_iXq_i)\xi,\xi) = (q_iXq_iU^{-1}\xi, U^{-1}\xi)$$

for all $X \in \mathcal{M}, \xi \in q_i'\tilde{\mathcal{H}}^+$. Now fix $X \in \mathcal{M}_{\text{s.a.}}$. Since $p_i'\tilde{\mathcal{M}}p_i'$ and $q_i'\tilde{\mathcal{M}}q_i'$ are $*$-isomorphic, there exists a unique operator $Y_i \in p_i'\tilde{\mathcal{M}}_{\text{s.a.}}p_i'$ such that $Y_i|_{q_i'\tilde{\mathcal{H}}} = \alpha_i(q_iXq_i)$. Using an isometry between the Jordan algebras, one sees that $\{\alpha_i(q_iXq_i)\}$ is bounded, because $\| \alpha_i(q_iXq_i) \| = \| q_iXq_i \| \leq \| X \|, i \in \mathrm{I}$. Thus $\{Y_i\}$ is bounded. We may then say that $\{Y_i\}$ converges to some operator $Y \in \tilde{\mathcal{M}}_{\text{s.a.}}$ in the σ-weak topology. We then have for $\xi \in \tilde{\mathcal{H}}^+$,

$$\begin{aligned}(Yq_j'\xi, q_j'\xi) &= \lim_i (Y_iq_j'\xi, q_j'\xi) = \lim_i (\alpha_i(q_iXq_i)q_j'\xi, q_j'\xi) \\ &= \lim_i (q_iXq_iU^{-1}q_j'\xi, U^{-1}q_j'\xi) \\ &= (XU^{-1}q_j'\xi, U^{-1}q_j'\xi).\end{aligned}$$

Taking a limit with respect to j, we obtain

$$(Y\xi,\xi) = (XU^{-1}\xi, U^{-1}\xi)$$

for all $\xi \in \tilde{\mathcal{H}}^+$. It is known that any normal state on the von Neumann algebra $\tilde{\mathcal{M}}$ is represented by a vector state with respect to an element of $\tilde{\mathcal{H}}^+$ (see [2, Lemma 2.10 (1)]). Therefore, the above element Y is uniquely determined. Moreover, we have $q_i'Yq_i' = \alpha_i(q_iXq_i)$. It follows that $\{\alpha_i(q_iXq_i)\}$ converges to Y in the strong topology. Hence one can define $\alpha(X) = Y$ for all $X \in \mathcal{M}$. It is now immediate that $\alpha(X^2) = \alpha(X)^2$ for all $X \in \mathcal{M}_{\text{s.a.}}$. Considering the inverse order isomorphism U^{-1}, we have $\alpha(\mathcal{M}) = \tilde{\mathcal{M}}$. This completes the proof. □

In the following lemma we deal with a reduced matrix-ordered standard form by a completely positive projection.

Lemma 2. *With $(\mathcal{M}, \mathcal{H}, \mathcal{H}_n^+)$ a matrix-ordered standard form, let E be a completely positive projection on $\mathcal{H}$. Then $(E\mathcal{M}E,\ E\mathcal{H},\ E_n\mathcal{H}_n^+)$ is a matrix-ordered standard form.*

Proof. The statement was shown in [8, Lemma 3] where $\mathcal{M}$ is σ-finite. In the case where $\mathcal{M}$ is not σ-finite, since E is a completely positive projection, there exists a von Neumann algebra $\mathcal{N}$ such that $(\mathcal{N},\ E\mathcal{H},\ E_n\mathcal{H}_n^+)$ is a matrix-ordered standard form by [6, Lemma 3]. Hence $E\mathcal{M}|_{E\mathcal{H}} = \mathcal{N}$ and $(E\mathcal{M}E,\ E\mathcal{H},\ E_n\mathcal{H}_n^+)$ is a matrix-ordered standard form by using the same discussion as in the proof in [7]. □

Now, we shall state the decomposition theorem for an order isomorphism between noncommutative L^2-spaces.

Theorem 3. *Let $(\mathcal{M}, \mathcal{H}, \mathcal{H}_n^+)$ and $(\tilde{\mathcal{M}}, \tilde{\mathcal{H}}, \tilde{\mathcal{H}}_n^+)$ be matrix-ordered standard forms. Suppose that A is a 1-positive map of $\mathcal{H}$ into $\tilde{\mathcal{H}}$ such that $A\mathcal{H}^+$ is a self-dual cone in the closed range of A. If both the support projection E and the range projection F of A are completely positive, then there exists a central projection P of $E\mathcal{M}E$ such that AP is completely positive and $A(E-P)$ is completely co-positive.*

In particular, if A is an order isomorphism of $\mathcal{H}$ onto $\tilde{\mathcal{H}}$, then there exists a central projection P of $\mathcal{M}$ such that AP is completely positive and $A(1-P)$ is completely co-positive.

Proof. We first consider the case where A is an order isomorphism. Let U, B and α be as in Proposition 1. It follows from a theorem of Kadison [5] that there exists a central projection P of $\mathcal{M}$ satisfying

$$\alpha : \mathcal{M}_P \to \tilde{\mathcal{M}}_{\alpha(P)}, \text{ onto } *\text{-isomorphism}$$

and

$$\alpha : \mathcal{M}_{1-P} \to \tilde{\mathcal{M}}_{\alpha(1-P)}, \text{ onto } *\text{-anti-isomorphism}.$$

Indeed, $\alpha(P)$ is a central projection of $\tilde{\mathcal{M}}$. Since α preserves a $*$-operation and power, $\alpha(P)$ is a projection. Suppose that Q is an arbitrary projection in $\mathcal{M}$. Since α is order preserving, we have $\alpha(QP) \leq \alpha(P)$ and $\alpha(Q(1-P)) \leq \alpha(1-P)$. It follows that two projections $\alpha(P)$ and $\alpha(QP)$ are commutative, and so are $\alpha(1-P)$ and $\alpha(Q(1-P))$. Hence, $\alpha(Q) = \alpha(QP + Q(1-P))$ and $\alpha(P)$ commute. Since α is bijective, a set $\alpha(Q)$ generates a von Neumann algebra $\tilde{\mathcal{M}}$. Therefore, $\alpha(P)$ belongs to a center of $\tilde{\mathcal{M}}$. Now, there then exists a unique completely positive isometry $u : P\mathcal{H} \to \alpha(P)\tilde{\mathcal{H}}$ such that

$$u(P\mathcal{H}^+) = \alpha(P)\tilde{\mathcal{H}}^+) \quad \text{and} \quad \alpha(x) = uxu^{-1}, \quad x \in \mathcal{M}_P$$

by [7, Proposition 2.4], which is also valid for the non-σ-finite case. Hence $(UxU^{-1}\xi, \xi) = (uxu^{-1}\xi, \xi), x \in \mathcal{M}_P, \xi \in \alpha(P)\tilde{\mathcal{H}}^+$. We have from the unicity of a completely positive isometry that $UP = u$. Note that $\alpha(P)UP = UP$. Indeed, we have for $\xi \in \alpha(1-P)\tilde{\mathcal{H}}^+$ the equality

$$\| PU^{-1}\xi \|^2 = (UPU^{-1}\xi, \xi) = (\alpha(P)\xi, \xi) = 0.$$

This yields $PU^{-1}\alpha(1-P) = O$, and so $PU^{-1} = PU^{-1}\alpha(P)$. Therefore, we obtain that $AP = UBJBJP = uBJBJP$ and AP is completely positive.

We next consider a $*$-isomorphism $\alpha' : \mathcal{M}_{1-P} \to \tilde{\mathcal{M}}'_{1-\alpha(P)}$ defined by $\alpha'(X) = \tilde{J}\alpha(X)^*\tilde{J}, X \in \mathcal{M}_{1-P}$. There then exists a unique completely positive isometry $v : (1-P)\mathcal{H} \to \alpha(1-P)\tilde{\mathcal{H}}$ such that

$$v(1-P)\mathcal{H}^+ = (1-\alpha(P))\tilde{\mathcal{H}}^+ \quad \text{and} \quad \alpha'(x) = vxv^{-1}, \quad x \in \mathcal{M}_{1-P}.$$

Then we have $\alpha(x) = \tilde{J}vx^*v^{-1}\tilde{J}, x \in \mathcal{M}_{1-P}$. Note that the complete positivity above means $v_n(1-P)_n\mathcal{H}_n^+ = (1-\alpha(P))_n\tilde{\mathcal{H}}_n^{+\prime}$, where $\tilde{\mathcal{H}}_n^{+\prime}$ denotes the self-dual cones associated with $\tilde{\mathcal{M}}'$ (cf. [10]). Hence v is a completely co-positive map under the setting $(\mathcal{M}, \mathcal{H}, \mathcal{H}_n^+)$ and $(\tilde{\mathcal{M}}, \tilde{\mathcal{H}}, \tilde{\mathcal{H}}_n^+)$. Hence

$$\begin{aligned}(UxU^{-1}\xi, \xi) &= (\tilde{J}vx^*v^{-1}\tilde{J}\xi, \xi) \\ &= (\tilde{J}\xi, vx^*v^{-1}\tilde{J}\xi) \\ &= (vxv^{-1}\xi, \xi)\end{aligned}$$

for all $x \in \mathcal{M}_{1-P}, \xi \in (1-P)\mathcal{H}^+$. It follows that $U(1-P) = v$. We conclude by the equality $A(1-P) = vBJBJ(1-P)$ that $A(1-P)$ is completely co-positive.

We now consider a general A. Since $A\mathcal{H}^+ \subset \tilde{\mathcal{H}}^+$, we have $A\mathcal{H}^+ \subset F\tilde{\mathcal{H}}^+$. Since F is a projection, $F\tilde{\mathcal{H}}^+$ is a self-dual cone in $F\tilde{\mathcal{H}}$. It follows from the self-duality of $A\mathcal{H}^+$ that $A\mathcal{H}^+ = F\tilde{\mathcal{H}}^+$. This yields from Lemma 2 that FAE is an order isomorphism of $E\mathcal{H}$ onto $F\tilde{\mathcal{H}}$ in the sense of matrix-ordered standard forms $(E\mathcal{M}E, E\mathcal{H}, E_n\mathcal{H}_n^+)$ and $(F\tilde{\mathcal{M}}F, F\tilde{\mathcal{H}}, F_n\tilde{\mathcal{H}}_n^+)$. Using the first part of the proof, we obtain the desired result. Indeed, there exists a central projection $P \in E\mathcal{M}E$

such that FAP is completely positive and $FA(E-P)$ is completely co-positive under the reduced matrix-ordered standard forms. We obtain the inclusion

$$^t(A_n(E_n - P_n)\mathcal{H}_n^+) = {}^t(F_nA_n(E_n - P_n)\mathcal{H}_n^+) \subset F_n\tilde{\mathcal{H}}_n^+ \subset \tilde{\mathcal{H}}_n^+.$$

This completes the proof. □

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