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原子力発電の経済性についての数学的考察

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1 Introduction

In 1970's, Dr.Schumacher appealed the abolition of atomic power generations because of various poisonous influences due to radiation [2], and in fact, even at the present day we have not found any safe disposal of the radio-active wastes. In this paper we shall innovate a mathematical model for the economy of atomic power generations and conduct Schumacher's appeal as a theorem. Moreover this theorem shows that the population of a society decreases under atomic power generations, which may be a kind of the paradox of the enrichment [1].

In the following, $x(t)$ denotes the population of the society at time t. First of all, when the society has no electric power generations, we shall assume that $x(t)$ obeys the logistic equation such that

$$\frac{dx}{dt} = a(1-x)\,x \qquad \text{(A)}$$

where a is a positive constant. Secondly, when the society has electric power generations which are not atomic, for example, hydro-dynamic or thermal dynamic, we may assume, by setting $y(t)$ to be the total energy of these non-atomic powers, that $x(t)$ and $y(t)$ obey the system such that

$$\begin{aligned} \frac{dx}{dt} &= a(1-x)\,x + y \\ \frac{dy}{dt} &= x - by \end{aligned} \qquad \text{(B)}$$

where a and b are positive constants. The meaning of (B) is the following : the first equation shows that the growth rate of $x(t)$ is excited by the addition of $y(t)$ to the society, where the unit of the power is appropriately chosen. Generally the electric power generation companies get their income

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from the society and pay the cost for the generation of the power, and this manner is described by the second equation, where the income is represented by $x(t)$ for appropriately chosen unit and the cost by $by(t)$, where the cost are , for example, for the maintenance of the dam in the hydro-electric power generation and for the fossil fuels in the thermal one. As is stated in Theorem 1, the system (B) has a stable equilibrium point, which may claim that our society is able to coexist with the non-atomic power generations.

Now we shall consider the case where the society has atomic power generations besides the non-atomic power generations as the case of our present day. Setting $z(t)$ to be the total energy of these atomic powers, we may assume that $x(t)$, $y(t)$ and $z(t)$ obey the following system

$$\begin{aligned}
\frac{dx}{dt} &= a(1-x)\,x + y + z \\
\frac{dy}{dt} &= \theta x - by \\
\frac{dz}{dt} &= (1-\theta)\,x - cz - d\int_0^t z(s)\,ds
\end{aligned} \tag{C}$$

where a, b, c, d and θ are positive constants and $0 < \theta < 1$. The meaning of (C) is the following. First of all, the number $t = 0$ of the lower extreme value of the integral of the third equation denotes the initial time for the atomic power generations to drive, and hence our initial condition is (1)

$$x(0) > 0 \quad y(0) > 0 \quad z(0) = 0 \tag{1}$$

The first equation shows that the growth rate of $x(t)$ is excited by the addition of $z(t)$ besides $y(t)$ to the society. The terms θx and $(1-\theta)x$ of the second and the third equations show the divided incomes of $x(t)$ into the two departments of the non-atomic power generations and the atomic power ones respectively by the electric power generation companies. Moreover, in the third equation, cz represents the cost for the uranic fuels for the atomic power generations and the term of the integral the cost of the treatment for the accumulated radio-active wastes from the initial time $t = 0$ to present time t, which is a feature of the atomic power generations.

2 Results

First of all, we shall treat system (B). The following Theorem 1 holds.

Theorem 1
The system (B) has the nontrivial equilibrium point $P = (x, y)$ such that $x = 1 + \frac{1}{ab}$ and $y = \frac{1}{b}(1 + \frac{1}{ab})$, which is asymptotically stable.

Remark 1
The stability of P may claim that our society is able to coexist with non-atomic powers generations. The proof of the stability of P follows from the standard arguments about the eigenvalues of the linear variational equation of the right hand side of(B) around P, and hence is omitted here.

Next we shall treat the system (C). From initial condition (1) we may verify that $x(t) > 0$, $y(t) > 0$ and $z(t) > 0$ for $0 < t < \omega$, where ω may be infinite. Then Theorem 2 holds,whose proof is stated in the next section 3.

Theorem 2
The following (i) and (ii) hold

(i) *If ω is finite, then $x(t)$, $y(t)$ and $z(t)$ are defined for $t = \omega$, and $z(\omega) = 0$, while $x(\omega) > 0$ and $y(\omega) > 0$.*

(ii) *If ω is infinite, then*

$$\begin{aligned} \lim_{t\to\infty} z(t) &= 0 \\ \lim_{t\to\infty} x(t) &= x_1 := 1 + \frac{\theta}{ab} \\ \lim_{t\to\infty} y(t) &= \frac{\theta}{b} x_1 \\ \int_0^\infty z(s) &= \Big(\frac{1-\theta}{d}\Big) x_1 \end{aligned}$$

Remark 2
The conclusion such that either $z(\omega) = 0$ or $\lim_{t\to\infty} z(t) = 0$ may claim the abolition of the atomic power generations. Moreover it is noted that x_1 is smaller than the x-component of the equilibrium of (B), $1 + \frac{1}{ab}$, which may claim the paradox of the enrichment [1].

We shall state the proof of Theorem 2 in the section 3 and several remarks about Theorem 2 in the section 4 respectively.

3 Proof of Theorem 2

First of all we shall prove (i). Since ω is finite, it follows from the second equation that

$$y(\omega) = e^{-b\omega} y(0) + \theta \int_0^\omega e^{-b(\omega-s)} x(s)\, ds$$

Since $x(s) > 0$ for $0 < s < \omega$, this implies that $y(\omega) > 0$. On the other hand, if $x(\omega) = 0$, then it follows from the first equation that $\frac{dx(\omega)}{dt} \geq y(\omega) >$

0, which implies that $x(t) < 0$ for immediately smaller than ω.
This is a contradiction to the definition of ω, and hence $x(\omega) > 0$. Therefore $z(\omega) = 0$.

Next we shall prove (ii). Since ω is infinite, $x(t) > 0$, $y(t) > 0$ and $z(t) > 0$ for $t > 0$. Firstly we shall show that $x(t)$, $y(t)$ and $z(t)$ are bounded for $t > 0$. Setting $w(t) = y(t) + z(t)$, we shall show that $x(t)$ and $w(t)$ are bounded for $t > 0$. Let consider the curve $(x(t), w(t))$ in the $x - w$ plane and the domains D_1 and D_2 in the $x - w$ plane such that

$$\begin{aligned} D_1 &= \{(x, w); 0 \leq x \leq x_1, \quad w \leq w_1\} \\ D_2 &= \{(x, w); x^2 + w^2 \leq R^2, \quad x \geq x_1\} \end{aligned}$$

where $x_1 = 1 + \frac{1}{a\alpha}$, $\alpha = min\{b, c\}$, $w_1 \geq \frac{x_1}{\alpha}$ and $R^2 = x_1^2 + w_1^2$. (see Fig.1)

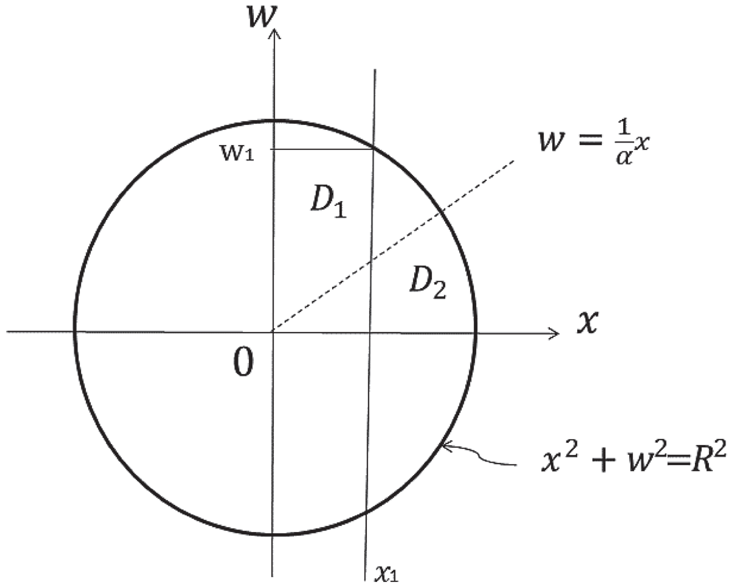

Figure 1:

We shall show that the curve $(x(t), w(t))$ crosses the boundary of $D_1 \cup D_2$, that is, L and C, where L is the straight line such that $\{w = w_1, 0 \leq x \leq x_1\}$ and C the part of the circle such that $\{x^2 + w^2 = R^2, x \geq x_1, w \geq 0\}$, from the outside into the inside, as t increases. In fact

$$\frac{dw}{dt} = \frac{dy(t)}{dt} + \frac{dz(t)}{dt} = x - by - cz - d\int_0^t z(s)\,ds$$

and hence

$$\frac{dw(t)}{dt} < x - by - cz \leq x - \alpha w$$

Therefore $\frac{dw(t)}{dt} < 0$ on L, which shows that the curve $(x(t), w(t))$ crosses L from the above to the below as t increases. (see Fig.2)

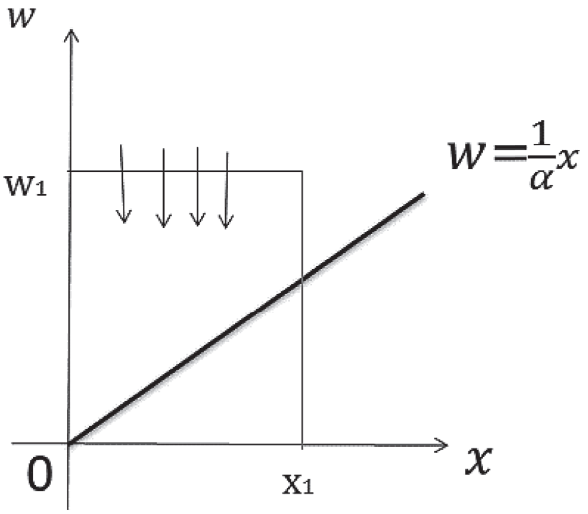

Figure 2:

Moreover, setting $V(t) = x^2(t) + w^2(t)$, we may obtain that

$$\frac{dV(t)}{dt} = 2x\frac{dx}{dt} + 2w\frac{dw}{dt} < 2x\{a(1-x)x + w\} + 2w\{x - \alpha w\}$$

and hence

$$\frac{dV(t)}{dt} < 2\left\{a(1-x) + \frac{1}{\alpha}\right\}x^2 - 2\left(\sqrt{\alpha}w - \frac{x}{\sqrt{\alpha}}\right)^2$$

Therefore $\frac{dV(t)}{dt} < 0$ for $x \geq x_1$, which shows that the curve $(x(t), w(t))$ crosses C from the outside into the inside, as t increases. (see Fig.3)

Above all $(x(t), w(t))$ crosses the boundary of $D_1 \cup D_2$ from the outside into the inside, as increases. Since $(x(0), w(0))$ is contained in $D_1 \cup D_2$ for large w_1, $(x(t), w(t))$ must remain in $D_1 \cup D_2$ for $t > 0$, which implies the boundedness of $x(t)$ and $w(t)$ for $t > 0$, that is, the boundedness of $x(t), y(t)$ and $z(t)$ for $t > 0$.

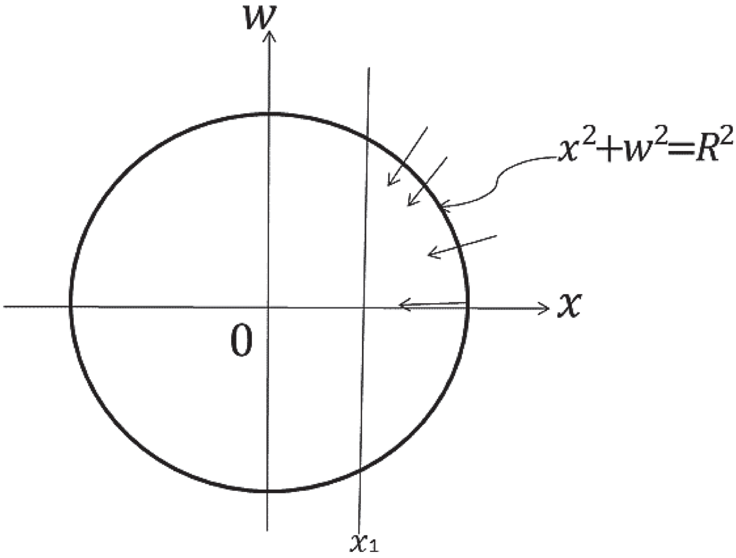

Figure 3:

Finally we shall show that $x(t), y(t)$ and $z(t)$ converge to their respective limits, as t goes to infinity. Setting

$$u(t) = \int_0^t z(s)\,ds$$

we may see that $u(t)$ satisfies the equation

$$\frac{d^2u}{dt^2} + c\frac{du}{dt} + du = (1-\theta)x(t)$$

Since $x(t)$ is bounded for $t > 0$, it follows that $u(t)$ is bounded for $t > 0$, that is, $\int_0^t z(s)\,ds$ is bounded for $t > 0$. Therefore the third equation implies that $\frac{dz(t)}{dt}$ bounded for $t > 0$, and hence $z(t)$ is uniformly continuous for $t > 0$. Since $z(t) > 0$ for $t > 0$, the boundedness of this integral above all implies that $\lim_{t\to\infty} z(t) = 0$. Moreover the third equation implies also that

$$\frac{d^2z}{dt^2} = (1-\theta)\frac{dx}{dt} - c\frac{dz}{dt} - dz$$

and hence $\frac{d^2z}{dt^2}$ is bounded for $t > 0$. Then we may see that $\lim_{t\to\infty} \dfrac{dz(t)}{dt} = 0$. In fact, if this is not true, there is a sequence $\{t_k\}$ such that t_k goes to infinity as k goes to infinity and that $|\frac{dz(t_k)}{dt}| > \varepsilon$ for some positive constant ε. Since $\frac{d^2z}{dt^2}$ is bounded, there is a positive constant δ such that $|\frac{dz(t)}{dt}| > \frac{\varepsilon}{2}$ for $|t - t_k| < \delta$, which contradicts to $\lim_{t\to\infty} z(t) = 0$. Since $\lim_{t\to\infty} \dfrac{dz(t)}{dt} = 0$, it follows from the third equation that $x(\infty) := \lim_{t\to\infty} x(t)$ exists and is equal to $\frac{d}{(1-\theta)}\int_0^\infty z(s)\,ds$. From the first equation, the same argument implies that $\lim_{t\to\infty}\dfrac{dx(t)}{dt} = 0$ and, hence that $y(\infty) := \lim_{t\to\infty} y(t)$ exists and $y(\infty) = -a(1-x(\infty))x(\infty)$, and moreover, from the second equation, $\lim_{t\to\infty}\dfrac{dy(t)}{dt} = 0$ and hence $y(\infty) = \frac{\theta}{b}x(\infty)$, which implies that $x(\infty) = 1 + \frac{\theta}{ab}$. Thus the proof of (ii) is completed.

4 Remarks

First of all. system (C) may be written as in the following 4-system

$$
\begin{aligned}
\frac{dx}{dt} &= a(1-x)\,x + y + z \\
\frac{dy}{dt} &= \theta x - by \\
\frac{dz}{dt} &= (1-\theta)\,x - cz - du \\
\frac{du}{dt} &= z
\end{aligned}
$$

where $z(0) = u(0) = 0$. This system has the equilibrium point $P(x_1, y_1, 0, u_1)$

$$x_1 = 1 + \frac{\theta}{ab}, \quad y_1 = \frac{\theta}{b}x_1, \quad u_1 = \Big(\frac{1-\theta}{d}\Big)x_1$$

We may see from [3] that P is asymptotically stable if d is sufficiently large.

Secondly our natural question is to determine whether ω is finite or infinite, and it seems that the saturation term $a(1-x)x$ of the first equation is too rigid to this question. Therefore we shall consider to relax this saturation term to more general function $f(x)$, while the result of theorem 2 almost holds, and hence treat instead of (C) the following system (D) such that

$$
\begin{aligned}
\frac{dx}{dt} &= f(x) + y + z \\
\frac{dy}{dt} &= \theta x - by \\
\frac{dz}{dt} &= (1-\theta)\,x - cz - d\int_0^t z(s)\,ds
\end{aligned}
\tag{D}
$$

where $z(0) = 0$. By the same argument as in the section 2, we can prove the following theorem, where ω is the same number as in Theorem 2.

Theorem 3
Assume that $f(x)$ is continuous for $x, f(0) = 0$ and $f(x) + \frac{x}{\alpha} < 0$ for large $x > 0$, where $\alpha = min\{b, c\}$. Then the following (i) and (ii) hold

(i) *If ω is finite, then $z(\omega) = 0$, while $x(\omega) > 0$ and $y(\omega) > 0$.*

(ii) *If ω is infinite, then $\lim_{t\to\infty} z(t) = 0$ and there exist $\lim_{t\to\infty} x(t)$, $\lim_{t\to\infty} y(t)$ and $\int_0^\infty z(s)\,ds$.*

In the following we assumed that $\theta = \frac{1}{2}, b = 1$ and $0 < c < 1$ in (D), which means that the electric company of the power generation divides the total income $x(t)$ equally to the both departments of the non-atomic power and the atomic power and that the cost of uranium fuels is taken to be cheaper than the one of the fossil fuels.

Example 1
We shall show the existence of $f(x)$ for the case where ω is finite. Assuming that $z(t) = 2t - t^2$ for $0 \le t \le 2$, where $\omega = 2$, we may obtain from the third equation that $x(t) = 4 - 4(1-c)t - 2ct^2 + 2dt^2 - \frac{2}{3}dt^3$. Since $\frac{dx(t)}{dt} = 4c - 4 + 4(d-c)t - 2dt^2$, it follows that $x(t) > 0$ and $\frac{dx(t)}{dt} < 0$ for $0 \le t \le 2$ in the case where $c < \frac{\sqrt{3}}{2}$ and $\frac{3}{2} < d < 1 + \sqrt{1-c^2}$, and hence in this case, for each of [0, 2] there corresponds one and only one x such that $x = x(t)$ for $x(2) \le x \le x(0)$, where $x(2) = \frac{8d}{3} - 4$ and $x(0) = 4$, and hence t is taken to be an function of x, say $t = t(x)$ for $x(2) \le x \le x(0)$. From the second equation we may obtain $y(t)$ such that $y(t) > 0$ for $0 \le t \le 2$ if $y(0) > 0$, and from the first equation

$$f(x) = \frac{dx(t)}{dt} - y(t) - z(t)$$

Here it is noted that the right hand side may be taken to be the function of x by the substitution of $t = t(x)$, while $f(x)$ is not determined for $0 < x < x(2)$ and for $x > x(0)$.

Example 2
We shall show the existence of $f(x)$ for the case where ω is infinite. Assuming that $z(t) = (\alpha t + \beta t^2)e^{-t}$ for $t \ge 0$, where α and β are positive constants, we may obtain from the third equation that

$$x(t) = 2d(\alpha+2\beta) - 2e^{-t}\{2\beta d + \alpha d - \alpha + (\alpha d + \alpha - \alpha c + 2\beta d - 2\beta)t + \beta(d+1-c)t^2\}$$

Since

$$\frac{dx(t)}{dt} = 2e^{-t}\{2\beta - 2\alpha + \alpha c + (\alpha d + \alpha - \alpha c + 2\beta c - 4\beta)t + \beta(d+1-c)t^2\}$$

it follows that $\frac{dx(t)}{dt} > 0$ for $t > 0$ in case where $2d > c^2 - 2c + 2$ and β is sufficiently large. Therefore in this case, for each $t \ge 0$ there corresponds *one and only one* $x = x(t)$ *for* $x(0) \leq x \leq x(\infty)$, *where* $x(0) = 2\alpha$ *and* $x(\infty) = 2d(\alpha + 2\beta)$, *and hence* t *is take to be the function of* x, *say* $t = t(x)$. *From the second equation we may obtain that* $y(t) > 0$ *for* $y(0) > 0$, *and from the third equation that* $f(x) = \frac{dx(t)}{dt} - y(t) - z(t)$, *which may be taken to be the function of* x *by the substitution* $t = t(x)$, *while* $f(x)$ *is not determined for* $0 < x < x(0)$ *and for* $x > x(\infty)$.

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